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Abstract

Full Text

MATHEMATICS

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ON THE DEPENDENCE ON A PARAMETER OF AN INTEGRAL OPERATOR

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1. Consider a system of ordinary differential equations with deviating argument

$$\frac{dx}{dt} = f\left[t, \lambda(t), x(t-h_1(t)), \dots, x(t-h_k(t))\right]; \tag{1}$$

here $x(t)$ is a vector-function with values in R^m. We shall assume that the right-hand sides of system (1) and the deviations $h_i(t)$, $i = 1, \dots, k$, possess the property of ω-periodicity in t,

$$f(t+\omega, x_1, y_1, \dots, y_k) \equiv f(t, x, y_1, \dots, y_k),$$

$$h_i(t+\omega) = h_i(t), \qquad i = 1, \dots, k,$$

where $-\infty < t < \infty,\ x \in R^m,\ y_j \in R^m,\ j = 1, \dots, k$.

M. A. Krasnosel'skii proposed in ([1]) a method for proving existence theorems for periodic solutions of systems (1), using reduction to special nonlinear integral equations whose right-hand sides depend on an auxiliary parameter λ. The operator defining the mentioned integral equations has the form

$$A(x;\lambda) = \int_0^t f\left[s, x(s), \tilde{x}_\omega(s-h_1(s,\lambda)), \dots, \tilde{x}_\omega(s-h_k(s,\lambda))\right] ds, \tag{2}$$

where $\tilde{x}_\omega$ is the ω-periodic extension of the function $x(t)$ from the interval $(0, \omega]$ to the whole axis.

With respect to the functions $h_i(s,\lambda)$ $(0 \le s \le \omega,\ 0 \le \lambda \le 1,\ i = 1, \dots, k)$ we assume that, for each fixed $\lambda_0 \in [0,1]$, they are measurable in s and, as $\lambda \to \lambda_0$, converge in measure to the functions $h_i(s,\lambda_0)$.

We shall assume that the vector-function $f(t, x, y_1, \dots, y_k)$ is continuous in the aggregate of variables $(t, x, y_1, \dots, y_k)$ $(t \in [0,\omega],\ x \in R^m,\ y_i \in R^m,\ i = 1, \dots, k)$.

Then, for each fixed λ_0, the operator (2) acts in the space C of vector-functions continuous on $[0, \omega]$ and is continuous in x uniformly with respect to $\lambda_0 \in [0, 1]$.

It can be shown that the fixed points of the operator

$$B(x; \lambda) = x(\omega) + A(x; \lambda)$$

are ω-periodic solutions of the equation

$$\frac{dx}{dt} = f\left[t, x(t), x(t - h_1(t)), \dots, x(t - h_k(t))\right]. \tag{3}$$

For applying to equation (1) the alternative principle for proving existence theorems for ω-periodic solutions, proposed in ([1]), it is necessary that the operator (2) be continuous in the aggregate of variables (λ, x) $(\lambda \in [0, 1],\ x \in C)$. For this, as is easily seen, it is sufficient that the operator (2), for fixed x, be continuous in λ.

In the present paper we give conditions under which operator (2) is continuous with respect to the parameter λ. Then, with the aid of the assertions proved and the alternative principle, existence theorems are established for ω-periodic solutions for a certain class of equations (1).

2. Consider a vector-function $g(t, y)$ $(t \in [0, \omega],\ x \in R^{m_1})$, taking its values in R^{m_2}. Suppose that the operator

$$gy = g[t, y(t)] \tag{4}$$

acts from the space S_{m_1} into S_{m_2}, where S_m denotes the space of measurable vector-functions on $[0, \omega]$ with values in R^m.

We shall say that a point (t_0, x_0) of the topological product $[0, \omega] \times R^{m_1}$ has type τ, if there exists a set $K \subseteq R^{m_1}$ such that the intersection of K with the ball $T(x_0, \rho) = \{x \mid x \in R^{m_1},\ |x - x_0| \le \rho\}$ has interior points in R^{m_1} for every positive ρ, and

$$\lim_{y_n \in K;\ y_n \to y_1} g(t_0, y_n) = g(t_0, y_0).$$

Everywhere in what follows we shall assume that all points $(t, y) \in [0, \omega] \times R^{m_1}$ are points of type τ. Below, $\Gamma(t)$ denotes the set of points of the space R^{m_1} at which the function $g(t, x)$, for fixed t, has discontinuities.

Theorem 1. *Suppose that the following conditions are satisfied:*

1) *The sequence of vector-functions $y_n(t)$ $(n = 1, 2, \dots)$ converges in measure to the vector-function $y_0(t)$ $(0 \le t \le \omega)$.*

2) *The set* Ω *of points* $t \in [0,\omega]$ *for which* $y_0(t) \in \Gamma(t)$ *is measurable.*

3) *The equality holds*

$$\lim_{n\to\infty} \operatorname{mes}\Omega_n = \operatorname{mes}\Omega, \tag{5}$$

where $\Omega_n = \{t \mid t \in \Omega,\ y_n(t) = y_0(t)\}$ *(obviously,* Ω_n *is measurable).*

Then the sequence of vector-functions

$$g[t, y_1(t)],\ g[t, y_2(t)], \ldots \tag{6}$$

converges in measure to the vector-function $g[t, y_0(t)]$.

Below, by C_ω we denote the subspace of vector-functions $x(t) \in C$ satisfying the condition $x(0) = x(\omega)$.

Theorem 2. *Operator (2) is continuous in the aggregate of the variables* (λ, x) *for* $\lambda \in [0,1]$ *and* $x \in C_\omega$.

It follows from this theorem that the operator

$$D(x;\lambda) = \int_0^t f[s, x(s), \widetilde{x_\omega}(s - \lambda h_1(s)), \ldots, \tilde{x}(s - \lambda h_k(s))]\, ds,$$

where $h_i(s)$ $(i = 1, \ldots, k;\ s \in [0,\omega])$ are finite and measurable almost everywhere, acts in C and is continuous in the aggregate of the variables (λ, x) $(\lambda \in [0,1],\ x \in C_\omega)$.

The following example shows that the operator D, if considered on the whole space C, may fail to possess the property of continuity with respect to λ. For simplicity, consider the space of scalar functions continuous on $[0,\omega]$, and define the operator $A_0(x;\lambda)$ by the equality

$$A_0(x;\lambda) = \int_0^t \widetilde{x_\omega}[s - \lambda h(s)]\, ds; \tag{7}$$

here $h(s)$ is the ω-periodic extension of the function $\varphi(s) = 2s$ from the interval $(0,\omega]$ to the entire axis. Let $\lambda_0 = 1/2$, and let the function $x_0(t)$ be defined as follows:

$$x_0(t) = \begin{cases} 0, & \text{for } t \in \left[0, \dfrac{\omega}{2}\right], \\ t - \dfrac{\omega}{2}, & \text{for } t \in \left(\dfrac{\omega}{2}, \omega\right]. \end{cases}$$

It is not difficult to verify that for $\lambda \in \left(\dfrac{1}{8\omega}, \dfrac{1}{2}\right)$ the equality

$$\tilde{x}_{0\omega}\left(s-\frac{1}{2}h(s)\right)-\tilde{x}_{0\omega}(s-\lambda h(s))=\begin{cases}\omega\left(\frac{1}{2}-2\lambda\right), & \text{for } s=0,\\ \frac{\omega}{2}, & \text{for } s\in(0,\omega].\end{cases}$$

holds.

It follows from this that $A_0(x_0,\lambda_0)-A_0(x_0,\lambda)=\omega t/2$ $(0\le t\le\omega)$ for $\lambda\in(1/8\omega,\,1/2)$, i.e., the operator (7) does not have the property of continuity with respect to λ at $\lambda_0=1/2$.

3. Suppose that the functions $h_i(s)$ $(i=1,\dots,k)$ are finite and measurable almost everywhere. We introduce into consideration the family of functions $h_i(s,\lambda)$, $0\le\lambda\le1$, $0\le s\le\omega$, defined by the formula

$$h_i(s,\lambda)=\begin{cases}h_i(s), & \text{for } \lambda=1;\\ N(\lambda)\operatorname{sign}h_i(s), & \text{for } \lambda\in[0,1) \text{ and } |h_i(s)|\ge N(\lambda);\\ h_i(s), & \text{for } \lambda\in(0,1) \text{ and } |h_i(s)|<N(\lambda);\end{cases} \tag{8}$$

here $N(\lambda)=\operatorname{tg}\pi\lambda/2$. Obviously, for any fixed $\lambda\in[0,1]$ the functions $h_i(s,\lambda)$ are finite and measurable almost everywhere. One can show that, for almost every $s\in[0,\omega]$, $h_i(s,\lambda)$ are continuous in λ.

Theorem 3. *Let the functions $h_i(s,\lambda)$ $(i=1,\dots,k)$ be defined by equality (8).*

Then the operator (2) is continuous in the aggregate of the variables (x,λ) $(x\in C,\ \lambda\in[0,1])$.

It follows from Theorem 3 that in equation (1) one can introduce the parameter λ $(0\le\lambda\le1)$ in such a way that the operator (2) corresponding to this equation will be continuous in the aggregate of the variables (x,λ) $(x\in C,\ \lambda\in[0,1])$.

4. Consider the system of ordinary differential equations

$$\frac{dx}{dt}=f(t,x,x,\dots,x), \tag{9}$$

obtained from system (1) when $h_i(s)\equiv0$ $(i=1,\dots,k)$. Suppose that the solution $x(t)=q(t,x_0)$, $x(0)=x_0$, of the Cauchy problem for system (9) is unique and defined on $[0,\omega]$. The operator U, defined by the equality

$$Ux=q(\omega,x),$$

is called the translation operator along the trajectories of system (9) over the time ω.

Following M. A. Krasnosel' skii, we shall say that the right-hand side of system (9) is nondegenerate at "infinity" if there exists a positive number ρ_0 such that for all $\rho > \rho_0$, on the spheres $S_\rho = \{x \mid x \in R^n,\ |x| = \rho\}$, the vector field

$$\varphi x = x - Ux$$

does not vanish and has nonzero rotation.

From the results of work ([1]) and Theorem 3 it follows that

Theorem 4. *Suppose that, for arbitrary ω-periodic, measurable, and almost everywhere finite deviations $h_i(s)$, the ω-periodic solutions of equation (1) (if, of course, they exist) lie in the ball $|x| \le r$. Let system (9) be nondegenerate at "infinity."*

Then equation (1), for arbitrary $h_i(s)$ $(i = 1, \dots, k)$, has ω-periodic solutions in the ball $|x_i| \le r$.

From Theorem 4 one can obtain various criteria for the existence of ω-periodic solutions of system (1). We give one simple assertion.

Below, by l_k $(k = 1, \dots, m)$ we denote the components of the vector $l \in R^m$.

Theorem 5. Suppose that the right-hand sides of the system

$$\frac{dx_i}{dt} = f_i[t, x_1(t), \dots, x_m(t); x_1(t - h(t)), \dots, x_m(t - h(t))] \quad (i = 1, \dots, m) \quad (10)$$

satisfy the following conditions:

1) The functions $f_i(t, x_1, \dots, x_m; y_1, \dots, y_m)$ are continuous in the aggregate of their variables and are ω-periodic in t.

2) The function $h(t)$ is measurable, finite almost everywhere, and periodic in t with period ω.

3) For each $i = 1, \dots, m$, each of the functions $f_i(t, x_1, \dots, x_m; y_1, \dots, y_m)$, for all values of $(x_1, \dots, x_m; y_1, \dots, y_m)$, satisfies one of the conditions:

a) $f_i(t, x_1, \dots, x_m; y_1, \dots, y_m) \ge -\gamma$,

b) $f_i(t, x_1, \dots, x_m; y_1, \dots, y_m) \le \gamma$,

where $\gamma > 0$.

4) One of the two inequalities holds:

c) $x_i \operatorname{sign} f_i(t, x_1, \dots, x_m; y_1, \dots, y_m) > 0$,

d) $x_i \operatorname{sign} f_i(t, x_1, \dots, x_m; y_1, \dots, y_m) < 0$

for $i = 1, \dots, m$ and $x, y \in G_i$, where

$G_i = \{x \mid |x_k| \le r+\gamma\omega\ (k = 1, 2, \dots, i-1);\ |x_i| \le r;\ x_k \in (-\infty, \infty)\ (k = i+1, \dots, m)\}$.

5) The system

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_m; x_1, \dots, x_m) \quad (i = 1, \dots, m)$$

is nondegenerate at "infinity."

Then system (10) has at least one ω-periodic solution lying in the ball $|x| \le r + \gamma\omega$.

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CITED LITERATURE

1. M. A. Krasnosel' skii, DAN, **154**, No. 3 (1963).

Note: Figure translations are in progress. See original paper for figures.