



Soviet-era science, translated into English

MATHEMATICS

S. L. KRUSHKAL'

1964

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196401.18348>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

S. L. KRUSHKAL'

ON MAPPINGS QUASICONFORMAL ON THE AVERAGE

(Presented by Academician M. A. Lavrent'ev, 14 II 1964)

1. Let E^n be n -dimensional Euclidean space with a fixed coordinate system; $|x|$ is the length of the vector $x = (x_1, x_2, \dots, x_n) \in E^n$, and

$$y = f(x) = (f_1(x), \dots, f_n(x))$$

is a mapping of a domain $D \subset E^n$ onto a domain D' ; D and D' are bounded. We say that $f(x) \in W_n^{(1)}(D)$ if all $f_i(x)$ have generalized first derivatives in the sense of S. L. Sobolev ⁽¹⁾, summable to the power n in D . By mA we shall denote the n -dimensional Lebesgue measure of a set $A \subset E^n$, and by D_h the set of points of D whose distance from the boundary of D is not less than h , $h > 0$. Put

$$\lambda(x, f) = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial f_i}{\partial x_j} \right)^2, \quad J(x, f) = \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} = \det \left\| \frac{\partial f_i}{\partial x_j} \right\| \geq 0$$

and denote by $Q_{m,n}(K)$ the class of topological mappings $y = f(x) \in W_n^{(1)}(D)$ of the domain D onto D' for which

$$I_m(f) = \frac{1}{mD'} \int_D \left(\frac{[\lambda(x, f)]^{n/2}}{n^{n/2} J(x, f)} \right)^m J(x, f) dx \leq K^m, \quad m \geq 1. \quad (1)$$

The closure of the class $Q_{m,n}(K)$ with respect to uniform convergence inside D will be denoted by $\overline{Q}_{m,n}(K)$. It is clear that if $f(x) \in \overline{Q}_{m,n}(K)$, then $f(x) \in W_n^{(1)}(D)$; for $m > 1$, $\overline{Q}_{m,n}(K) \subset \overline{Q}_{1,n}(K)$, and for mappings of the class $Q_{1,n}(K)$ the inequality

$$I_1(f) = \frac{1}{mD'} \int_D [\lambda(x, f)]^{n/2} dx \leq n^{n/2} K. \quad (2)$$

holds.

In particular, if $n = 2$ and D and D' are unit disks, then we obtain the classes $Q_m(K)$ and $\overline{Q}_m(K)$, introduced by L. Ahlfors ⁽²⁾, of mappings $\zeta = f(z)$ of the

disk $|z| < 1$ onto the disk $|\zeta| < 1$, for which

$$I_m(\zeta) = \frac{1}{\pi} \iint_{|\zeta| < 1} \left(\frac{|z_\zeta|^2 + |z_{\bar{\zeta}}|^2}{|z_\zeta|^2 - |z_{\bar{\zeta}}|^2} \right)^m d\sigma_\zeta \leq K^m.$$

In ⁽²⁾ it is proved that mappings of the class $\overline{Q}_m(K)$ are equicontinuous for $m > 2$ inside the disk $|z| < 1$ and carry a measurable set A into a set $f(A)$ whose measure is equal to

$$mf(A) = \iint_A (|f_z|^2 - |f_{\bar{z}}|^2) d\sigma_z.$$

In this note we study some properties of mappings from $Q_{m,n}(K)$, $n \geq 2$.

Lemma 1 ^(3,4). *If the function $u(x)$ is continuous in*

$$R: \quad r_1 < |x| < r_2,$$

$u(x) \in W_n^{(1)}(D)$, and

$$\int_R [\lambda(x, u)]^{n/2} dx \leq M,$$

then

$$\int_{r_1}^{r_2} (\text{osc}_{|x|=r} u)^n \frac{dr}{r} \leq C_n \int_R [\lambda(x, u)]^{n/2} dx \leq MC_n,$$

where C_n is a constant depending only on n .

From Lemma 1 it follows that for every δ , $0 < \delta < 1$, there is an $r = r(\delta)$, $\delta \leq r \leq \sqrt{\delta}$, such that

$$\text{osc}_{|x|=r} u \leq \left(\frac{2MC_n}{\ln 1/\delta} \right)^{1/n}. \quad (3)$$

For $n = 2$ this assertion coincides with a well-known lemma of R. Courant ⁽⁵⁾.

Theorem 1. *Mappings of the class $\overline{Q}_{1,n}(K)$ are uniformly continuous on every compact set $B \subset D$.*

For $f(x) \in Q_{1,n}(K)$ the assertion of the theorem is obtained by applying Lemma 1, and for mappings of the class $\overline{Q}_{1,n}(K)$ by a subsequent passage to the limit.

Theorem 2. *If $y = f(x)$ is an open mapping of the class $\overline{Q}_{1,n}(K)$, then for every measurable set A the set $f(A)$ is measurable.*

Proof. Approximating the function $f(x)$ by a sequence of continuous piecewise-linear functions $f_k(x) \in W_n^{(1)}(D_h)$ (for fixed $h > 0$) converging to $f(x)$ uniformly on D_h , and such that

$$\|f_k(x) - f(x)\|_{W_n^{(1)}(D_h)} \rightarrow 0$$

as $k \rightarrow \infty$, one can show that for every open set $G \subset D$ one has

$$mf(G) \leq \chi \int_G J(x, f) dx.$$

Hence it follows that if A is an arbitrary set of measure zero in D , then $mf(A) = 0$, and this is equivalent to the assertion of the theorem.

For $n = 3$ the following stronger theorem is valid.

Theorem 3. *If $y = f(x)$ is an open mapping of the class $\overline{Q}_{1,3}(K)$, then for every measurable set A the set $f(A)$ is measurable and its Lebesgue measure is equal to*

$$mf(A) = \int_A J(x, f) dx.$$

The proof of Theorem 3 is based on Theorem 2 and the following lemma:

Lemma 2. *Let a continuous vector function $y = f(x) \in W_p^{(1)}(D)$, $p > 2$, realize an open mapping of a domain $D \subset E^3$. Then $f(x)$ has a complete differential almost everywhere in D .*

The proof of Lemma 2 relies on Theorem 1.3 of (7) and some considerations given in (9).

Theorem 4. *If $y = f(x) \in \overline{Q}_{m,n}(K)$, then $I_m(f) \leq K^n$.*

The assertion of the theorem is obtained by applying certain relations from (9), p. 221.

Denote by U_r the ball $|x| < r$.

Theorem 5. *There exists a universal function $\mu(\varepsilon, r)$, defined for $\varepsilon > 0$, $0 \leq r < 1$, with $\mu(\varepsilon, r) \rightarrow 0$ as $\varepsilon \rightarrow 0$, such that for every mapping $y = f(x) \in \overline{Q}_{m,n}(1 + \varepsilon)$ of the ball U_1 onto itself there exists a Möbius mapping $a(x)$ such that, for $|x| \leq r < 1$,*

$$|f(x) - a(x)| \leq \mu(\varepsilon, r).$$

This theorem is proved by the same method as the analogous assertions for the classes of mappings considered in (9, 10).

As is known, for q -quasiconformal mappings $y = f(x)$ of plane and spatial domains ($q < \infty$), the inverse mappings $x = f^{-1}(y)$ are also q -quasiconformal and uniformly continuous. It can be shown that the classes $\overline{Q}_{m,n}(K)$ no longer possess this property for any m , $1 \leq m < \infty$.

2. It is known that the family of topological mappings of the disk $|z| \leq 1$ onto the disk $|w| \leq 1$, $w(0) = 0$, with uniformly bounded Dirichlet integrals is uniformly continuous in the closed disk $|z| < 1^*$. Denote by $p(z)$ and $\theta(z)$

the characteristics of the quasiconformal mapping $w = f(z)$ in the sense of M. A. Lavrent'ev.

* This result is easily obtained, for example, by means of R. Courant's lemma; see also (12).

Theorem 6. Let the function $w = f(z)$ map quasiconformally the disk $|z| \leq 1$ onto the disk $|w| \leq 1$, with $f(0) = 0$, $f(1) = 1$, and

$$\iint_{|z| < 1} (p(z) - 1) d\sigma_z < \varepsilon.$$

Then, for all z , $|z| \leq 1$,

$$|f(z) - z| < \mu(\varepsilon), \quad (4)$$

where $\mu(\varepsilon)$ depends only on ε and $\lim_{\varepsilon \rightarrow 0} \mu(\varepsilon) = 0$.

Proof. Suppose that the theorem is false. Then there exist a sequence of mappings $w = f_k(z)$ and points z_k , $|z_k| \leq 1$ ($k = 1, 2, \dots$), such that

$$\iint_{|z| < 1} (p_k(z) - 1) d\sigma_z \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

but

$$|f_k(z_k) - z_k| \geq \varepsilon_0 > 0, \quad \varepsilon_0 = \text{const.} \quad (5)$$

Choosing a subsequence $z = f_{k_s}^{-1}(w) = \varphi_s(w)$ converging uniformly in the disk $|w| \leq 1$ to some function $z = \varphi_0(w)$, we shall have:

$$\begin{aligned} \iint_{|w| < 1} \left(\left| \frac{\partial \varphi_0}{\partial w} \right|^2 + \left| \frac{\partial \varphi_0}{\partial \bar{w}} \right|^2 \right) d\sigma_w &\leq \lim_{s \rightarrow \infty} \iint_{|w| < 1} \left(\left| \frac{\partial \varphi_s}{\partial w} \right|^2 + \left| \frac{\partial \varphi_s}{\partial \bar{w}} \right|^2 \right) d\sigma_w \leq \pi = \\ &= \iint_{|w| < 1} \left(\left| \frac{\partial \varphi_0}{\partial w} \right|^2 - \left| \frac{\partial \varphi_0}{\partial \bar{w}} \right|^2 \right) d\sigma_w, \end{aligned}$$

whence it follows that $\partial \varphi_0(w) / \partial \bar{w} = 0$ for almost all w , $|w| \leq 1$. Thus $\varphi_0(w)$ is an analytic function in the disk $|w| < 1$, continuous in the closed disk $|w| \leq 1$, assumes all values from the disk $|z| \leq 1$, satisfies $\varphi_0(w) \leq 1$, $\varphi_0(0) = 0$, and $\varphi_0(1) = 1$. Therefore $\varphi_0(w) = w^l$, where l is some natural number. But

$$\iint_{|w| < 1} |\varphi'_{0w}|^2 d\sigma_w = \pi,$$

and hence $l = 1$. Consequently, for any $\varepsilon > 0$, when $s \geq s_0(\varepsilon)$,

$$|\varphi_s(w) - w| < \varepsilon$$

for all w , $|w| \leq 1$, which contradicts (5). The theorem is proved.

This theorem is proved by another method in ⁽¹¹⁾.

Theorem 7. Let $w = f_0(z)$ be a q -quasiconformal mapping of the disk $|z| < 1$ onto the disk $|w| < 1$; let $w = f_k(z)$ be a sequence of topological mappings of the disk $|z| \leq 1$ onto $|w| \leq 1$ of class $Q_1(K)$, with

$$f_k(0) = 0, \quad f_k(1) = 1,$$

and

$$h_k(z) \rightarrow h_0(z)$$

in measure, where

$$h_k(z) = \left| \frac{\partial f_k}{\partial \bar{z}} \right| \bigg/ \left| \frac{\partial f_k}{\partial z} \right|$$

($k = 0, 1, 2, \dots$). Then the sequence $f_k(z)$ converges uniformly to $f_0(z)$ in the disk $|z| \leq 1$.

The proof of Theorem 7 is based on applying Green's formula to the functions $f_k[f_0^{-1}(w)]$.

In conclusion I express my deep gratitude to Prof. P. P. Belinskii and Prof. Yu. G. Reshetnyak for their advice and comments.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

Received
7 II 1964

References

1. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.
2. L. Ahlfors, *J. Anal. Math.*, **3**, 1 (1953-1954).
3. F. W. Gehring, *Trans. Am. Math. Soc.*, **103**, 3, 353 (1962).
4. M. A. Kreines, *Matem. sborn.*, **9** (51), 3, 713 (1941).
5. R. Courant, *Dirichlet's Principle, Conformal Mappings and Minimal Surfaces*, IL, 1953.
6. Yu. G. Reshetnyak, *Sibirsk. matem. zhurn.*, **3**, 5, 744 (1962).
7. B. V. Boyarskii, *Matem. sborn.*, **43** (85), 4, 459 (1957).
8. F. W. Gehring, O. Lehto, *Ann. Acad. Sci. Fenn., Ser. A 1*, No. 272 (1959).
9. Yu. G. Reshetnyak, in: *Some Problems of Mathematics and Mechanics*, Novosibirsk, 1961.
10. Yu. G. Reshetnyak, *DAN*, **152**, No. 2, 286 (1963).
11. P. P. Belinskii, *Uchen. zap. L'vovsk. gos. univ.*, **29**, 6, 58 (1954).
12. G. D. Suvorov, *Matem. sborn.*, **45** (87), 2, 176 (1958).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.