MATHEMATICS

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Abstract

Full Text

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ON THE SOLUTION OF NONLINEAR EQUATIONS BY THE METHOD OF ITERATIONS

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Let in a real Hilbert space H there be given a twice continuously differentiable functional $f(x)$ and let $F(x)$ be the gradient of the functional $f(x)$, i.e., the operator defined by the formula

$$\lim_{t\to 0}\frac{f(x+th)-f(x)}{t}=(F(x),h)$$

for arbitrary $h\in H$ (see, for example, (1)).

In this note we consider iterative methods for solving the equation

$$F(x)=0. \tag{1}$$

1. Theorem 1. Let there exist $x_0\in H$ and a number $r>0$ such that:

a) $(F'(x)h,h)\geq m(h,h)$, $m>0$ for $h\in H$ and $x\in S(x_0,r)$;

b) $r\geq \|F(x_0)\|/m$;

c) $(F'(x)h,h)\leq M(h,h)$ for $h\in H$ and $x\in S(x_0,\|F(x_0)\|/m)$.

Then in the sphere $S(x_0,\|F(x_0)\|/m)$ there exists a solution x^* of equation (1), unique in the sphere $S(x_0,r)$, and the iterative process starting from x_0 converges to it:

$$x_{n+1}=x_n-\frac{2}{M+m}F(x_n)\quad (n=0,1,2,...),$$

moreover

$$\|x_n-x^*\|\leq\left(\frac{M-m}{M+m}\right)^n\|x_0-x^*\|.$$

Remark. Under the conditions of Theorem 1, the solution x^* of the equation $F(x)=0$, unique in $S(x_0,r)$, is also the unique point of minimum of the functional $f(x)$ in $S(x_0,r)$.

In the case when the computations are carried out with errors, the following holds:

Theorem 2. Let there exist $x_0 \in H$ and numbers $r > 0$ and $\bar{h} > 0$ such that:

a) $(F'(x)h, h) \geq m(h, h)$, $m > 0$ for $h \in H$ and $x \in S(x_0, r)$;

b)

$$r \geq \frac{\|F(x_0)\| + \bar{h}}{m};$$

c) $(F'(x)h, h) \leq M_1(h, h)$ for $h \in H$ and

$$x \in S\left(x_0, \frac{1}{m}[\|F(x_0)\| + \bar{h}]\right).$$

Then for any sequence $h_n \in H$ with $\|h_n\| \leq \bar{h}$, the iterative process starting from x_0,

$$x_{n+1} = x_n - \frac{2}{M_1 + m}\left[F(x_n) + h_n\right] \quad (n = 0, 1, 2, ...)$$

gives a sequence x_n such that

$$\|x_n - x^*\| \leq \left(\frac{M_1 - m}{M_1 + m}\right)^n \|x_0 - x^*\| + \frac{\bar{h}}{m}.$$

2. **Theorem 3.** Suppose

$$(F'(x)h, h) \geq m(h, h), \qquad m > 0, \quad x, h \in H;$$

A_n is a sequence of symmetric, bounded, positive definite operators such that

$$\overline{m}(A_n^{-1}h, h) \leq (F'(x)h, h) \leq \overline{M}(h, h),$$

$$0 < \overline{m} \leq \overline{M} < \infty \quad \text{for } h \in H \text{ and } x \in S\left(x^*, \sqrt{\frac{2}{m}\left[f(x_0) - f(x^*)\right]}\right),$$

where x_0 is some element of H.
Then the function $\psi(\alpha) = f(x_n - \alpha A_{nF}(x_n))$ has a unique minimum α_n, and the iterative process, starting from x_0,

$$x_{n+1} = x_n - \gamma_n A_{nF}(x_n) \qquad (n = 0, 1, 2, ...),$$

where γ_n are arbitrary numbers satisfying the condition $\varepsilon \le \gamma_n \le \alpha_n,\ \varepsilon > 0$, gives a minimizing sequence for $f(x)$, with $f(x_n) \to f(x^*)$ and $x_n \to x^*$ at the rate of a geometric progression with ratios respectively τ and $\sqrt{\tau}$, where

$$\tau = 1 - \varepsilon_1 \left(1 - \frac{\varepsilon_1 \overline{M}}{2}\right) \frac{2\overline{m}^2}{M}, \qquad \varepsilon_1 = \min\left\{\varepsilon, \frac{1}{\overline{M}}\right\}.$$

In particular, the iterative process

$$x_{n+1} = x_n - \alpha_n A_{nF}(x_n) \qquad (n = 0, 1, 2, \ldots)$$

converges, starting from any x_0, at the rate of a geometric progression with ratio

$$\sqrt{1 - m^2/M^2}.$$

3. **Corollary 1.** Suppose

$$(F'(x)h, h) \ge m(h, h), \qquad m > 0, \quad x, h \in H$$

and A_n is a sequence of symmetric, bounded, positive definite operators, with

$$m_A(h, h) \le (A_{nh}, h) \le M_A(h, h), \qquad h \in H, \quad 0 < m_A \le M_A < +\infty.$$

Then the function $\psi(\alpha) = f(x_n - \alpha A_{nF}(x_n))$ has a unique minimum α_n, and the iterative process

$$x_{n+1} = x_n - \gamma_n A_{nF}(x_n) \qquad (n = 0, 1, 2, \ldots),$$

where $\varepsilon \le \gamma_n \le \alpha_n,\ \varepsilon > 0$, starting from any x_0, gives a minimizing sequence for $f(x)$, with $f(x_n) \to f(x^*)$ and $x_n \to x^*$ at the rate of a geometric progression.

Corollary 2. Suppose

$$(F'(x)h, h) \ge m(h, h), \qquad m > 0, \quad x, h \in H.$$

Then the iterative process

$$x_{n+1} = x_n - \gamma_n [F'(x_n)]^{-1} F(x_n) \qquad (n = 0, 1, 2, \ldots)$$

with $\varepsilon \le \gamma_n \le \alpha_n,\ \varepsilon > 0$, where α_n is the unique minimum of the function $f(x_n - \alpha[F'(x_n)]^{-1}F(x_n))$, gives a minimizing sequence for $f(x)$, with $f(x_n) \to f(x^*)$ and $x_n \to x^*$ at the rate of a geometric progression.

Theorem 4. Suppose the conditions of Theorem 3 are fulfilled and

$$\|A_n - [F'(x^*)]^{-1}\| \to 0 \qquad \text{as } n \to \infty.$$

Then the iterative process

$$x_{n+1} = x_n - \alpha_n A_{nF}(x_n) \qquad (n = 0, 1, 2, \dots)$$

converges, starting from any x_0, to x^* at a rate higher than a geometric progression, i.e.,

$$\|x_n - x^*\| \leq C q_1 q_2 \dots q_n, \qquad q_k < 1, \qquad q_k \to 0 \quad \text{as } k \to \infty.$$

Corollary 3. Let the condition

$$(F'(x)h, h) \geq m(h, h), \qquad m > 0, \qquad x, h \in H$$

be satisfied. Then the iterative process

$$x_{n+1} = x_n - \alpha_n [F'(x_n)]^{-1} F(x_n) \qquad (n = 0, 1, 2, \dots),$$

where α_n is determined from the condition that the function

$$\psi(\alpha) = f(x_n - \alpha [F'(x_n)]^{-1} F(x_n))$$

attain its minimum, converges to x^* at a rate greater than that of a geometric progression.

The equation considered in this paper is not the general equation $\Phi(x) = 0$, where $\Phi(x)$ is an operator from $\mathscr{H}$ into itself. The proofs of the assertions indicated above essentially use the fact that $F(x)$ is the gradient of a functional. In particular, it is essential that $F'(x)$ be a symmetric operator. In the finite-dimensional case of n dimensions, the problem considered is the problem of finding a stationary point.

The iterative process

$$x_{n+1} = x_n - \alpha_n [F'(x_n)]^{-1} F(x_n) \qquad (n = 0, 1, 2, \dots),$$

unlike Newton' s method, which for its convergence requires an initial approximation close to the solution x^*, converges from any initial approximation.

Unlike the method of steepest descent, the rate of convergence increases as the solution is approached.

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CITED LITERATURE

1. M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, Moscow, 1956.

Note: Figure translations are in progress. See original paper for figures.

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