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Abstract

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MATHEMATICS

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EXTENSION OF THE L-FOURIER TRANSFORM TO FUNCTIONS WITH LOCALLY INTEGRABLE SQUARE

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Let L be the operator generated in the Hilbert space $L^2(R^+)$, $R^+ = (0, \infty)$, by the differential expression $ly = -y'' + p(x)y$ and the boundary condition $y'(0) - \theta y(0) = 0$; $p(x)$ is a complex-valued function; θ is a complex number; L is a non-self-adjoint operator. Everywhere below we shall assume that one of the following two conditions is satisfied:

a) there exists an $\varepsilon > 0$ such that $p(x) \exp(\varepsilon x)$ is summable on R^+;

b) $p(x)$ is continuous on R^+ and admits an analytic continuation to the domain $\{x : x = \rho \exp(i\varphi),\ \rho > \rho_0,\ |\varphi| < \varphi_0\}$, $\rho_0 > 0,\ \varphi_0 > 0$, having order $O(|x|^{-a})$, $a > 3$, as $|x| \to \infty$.

In accordance with results of M. A. Naimark and the author (see ([1,2])), each of these conditions ensures the finiteness of the set of eigenvalues and spectral singularities of the operator L^*.

Let $\omega(x, \lambda)$ be the solution of the equation $ly = \lambda y$ satisfying the initial conditions $\omega(0, \lambda) = 1$, $\omega'_x(0, \lambda) = \theta$. Put

$$\omega(f, \lambda) = \int_0^\infty f(x)\omega(x, \lambda)\, dx. \tag{1}$$

The L-Fourier transform of a function $f \in L^2(R^+)$ is the quantity ωf, determined by the function $\omega(f, \lambda)$, $\lambda > 0$ (the half-axis $\lambda \geq 0$ is the continuous spectrum of the operator L) and by the numbers $\omega^{(j)}(f, \lambda_k)$, $j = 0, \dots, m_k - 1;\ k = 1, \dots, r$ ($\lambda_1, \dots, \lambda_r$ are the eigenvalues of the operator L; $m_1, \dots, m_r$ are their multiplicities), understood as the result of (formal) differentiation with respect to λ under the integral sign in (1).

If the operator L has spectral singularities, then the inversion formula for the L-Fourier transform contains the regularized value of a certain divergent integral and a "regularization correction" —a term having the form of a linear combination of the principal (i.e., proper and associated) functions of the operator L corresponding to the spectral singularities (see ([2])). These principal functions are not elements of $L^2(R^+)$, and therefore it is natural to try to extend the L-Fourier transform to functions that are not square-integrable. In the author's paper ([3]) the L-Fourier transform was extended from $L^2(R^+)$ to a certain class F of exponentially growing functions. The construction of the extension (carried out under the assumption that condition a) formulated above holds, but not condition b)) was based on the connection between the L-Fourier transform and the ordinary Fourier transform, established by means of the corresponding transformation operator. In this case the restriction on the order of growth of functions of the class F was caused by the necessity of operating only with those spaces of generalized functions in which the function**

$$A(s) = y'_{1x}(0, s) - \theta y_1(0, s) \tag{2}$$

($y_1(x, s)$ is the solution of the equation $ly = s^2 y$, which, for $\operatorname{Im} s > 0,\ x \to \infty$, is asymptotically equal to $\exp ixs$) is a multiplier. It turned out that the L-Fourier transform, extended to the class F, is not uniquely invertible; it is equal to zero on the subspace $\tilde{\omega}(L) \subset F$ spanned by the principal functions corresponding to the spectral singularities.

* Spectral singularities of the operator L are poles of the density of its spectral function, located on the continuous spectrum.

** The function $A(s)$ was introduced by M. A. Naimark in ([1]). If $A(s) = 0$ and $\operatorname{Im} s > 0$, then s^2 is an eigenvalue of the operator L; if $A(s) = 0$, $\operatorname{Im} s = 0$, $s \neq 0$, then s^2 is a point of the continuous spectrum of the operator L, called a spectral singularity.

In the present article another method is used for extending the Fourier L-transform. This method is more general, since it is also applicable in the case when, instead of condition a), condition b) is satisfied (see above). Moreover, this method makes it possible to extend the Fourier L-transform from the space $L^2(R^+)$ to the space $L^2_*(R^+) \supset F$. (By $L^2_*(R^+)$ we shall denote the space of functions with locally integrable square, i.e., functions whose square is integrable on every finite interval $(0, a) \subset R^+$.) This method is also simpler: it is based not on a connection with the ordinary Fourier transform, but on the corresponding generalized Parseval equality*.

Let us note that after the extension to the space $L^2_*(R^+)$, the set of functions for which the Fourier L-transform is equal to zero still coincides with the subspace $\tilde{\omega}(L)$ spanned by the principal functions of the spectral singularities. However, the nonunique invertibility of the Fourier L-transform does not hinder the successful application of this transform to the study of the corresponding boundary-value problems for partial differential equations. We shall consider this question elsewhere (see also the author' s paper ([3])), and now proceed to the description of the construction of the extension.

The Parseval equality corresponding to the operator L has the form

$$\int_0^\infty f(x)g(x)\,dx = \frac{1}{\pi}\int_0^\infty \frac{\omega(f,\lambda)\omega(g,\lambda)\sqrt{\lambda}}{A(\sqrt{\lambda})\,A(-\sqrt{\lambda})}\,d\lambda +$$

$$+\sum_{k=1}^{r}\left\{\left[\left(\frac{d}{d\lambda}\right)^{m_k-1} M_k(\lambda)\omega(f,\lambda)\omega(g,\lambda)\right]\right\}_{\lambda=\lambda_k}, \qquad (3)$$

where

$$M_k(\lambda) = (\lambda-\lambda_k)^{m_k} y_1(0,\sqrt{\lambda})/[(m_k-1)!\,A(\sqrt{\lambda})].$$

If the operator L has spectral singularities, and f and g are arbitrary functions from $L^2(R^+)$, then the integral on the right-hand side of (3), generally speaking, diverges. However, this integral converges and equality (3) is valid if at least one of the functions f or g belongs to the manifold

$$\mathfrak{H} = \left\{h:\ h\in L^2(R^+),\quad \int_{-\infty}^{\infty}\left|\frac{\omega(h,s^2)s}{A(s)}\right|^2 ds<\infty\right\},$$

and the other is an arbitrary element of $L^2(R^+)$. Put

$$L^2(R_a^+) = \{f:\ f\in L^2(R^+),\ f(x)=0 \text{ for } x>a\},$$

$$\mathfrak{H}(a) = \mathfrak{H}\cap L^2(R_a^+),\qquad \mathfrak{H}_0=\bigcup\mathfrak{H}(a),\quad 0<a<\infty.$$

We shall say that a sequence $\{f_n\}\subset\mathfrak{H}_0$ converges (in the sense of $\mathfrak{H}_0$) if there exists an $a>0$ such that $\{f_n\}\subset\mathfrak{H}(a)$ and this sequence converges in the norm of $L^2(R^+)$. It is not difficult to see that the space $\mathfrak{H}_0$ is the orthogonal complement in the space

$$L_0^2(R^+) = \bigcup L^2(R_a^+),\quad 0<a<\infty,$$

of the subspace spanned by the principal functions of the spectral singularities of the adjoint operator L^*. Since these principal functions are linearly independent modulo $L^2(R^+)$, it follows from this that $\mathfrak{H}_0$ is dense in $L^2(R^+)$. It is natural to take the space $\mathfrak{H}_0$ as the space of basic elements of the space $L^2(R^+)$. For each function $f^*\in L_*^2(R^+)$, by $\mathcal{P}f^*$ we shall denote the functional defined by the relation

$$\langle f,\mathcal{P}f^*\rangle = \int_0^\infty f(x)f^*(x)\,dx,\qquad f\in\mathfrak{H}_0.$$

Theorem 1. *In order that a linear functional f', defined on the space $\mathfrak{H}_0$, be continuous, it is necessary and sufficient that there exist a function $f^* \in L^2_*(R^+)$ such that $f' = \mathcal{P} f^*$. In this case $\mathcal{P} f^* = 0$*

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* The idea of the method used here arose for the author in connection with certain remarks of B. M. Levitan.

then and only then when $f^* \in \widetilde{\mathfrak{S}}(L)$, i.e. when

$$f^*(x) = \sum_{k=1}^{\rho} \sum_{j=0}^{\mu_k - 1} C_{kj} \omega^{(j)}(x, \tilde{\lambda}_k). \qquad (4)$$

Here $\tilde{\lambda}_1, \dots, \tilde{\lambda}_\rho$ are the spectral singularities of the operator L, $\mu_1, \dots, \mu_\rho$ are their multiplicities, and C_{kj} are arbitrary numbers.

Thus, the space of generalized elements $\mathfrak{H}_0'$ is isomorphic to the factor space $L^2(R^+)/\tilde{\omega}(L)$.

Denote by $\mathfrak{G}(a)$ the linear space of functions $\xi(\lambda)$, $|\lambda| < \infty$, such that:

1°. $\xi(s^2)$ is an entire function of s of finite degree $\leq a$, square-integrable on the axis $\operatorname{Im} s = 0$.

2°. $\xi^{(j)}(\tilde{\lambda}_k) = 0; \quad j = 0, \dots, \mu_k - 1; \quad k = 1, \dots, \rho.$

The following fact is of decisive importance for what follows:

Theorem 2. The space $\mathfrak{G}(a)$, endowed with the norm

$$|\xi| = \left\{ \int_{-\infty}^{\infty} \left| \frac{\xi(s^2) s}{A(s)} \right|^2 ds \right\}^{1/2} + \sum_{k=1}^{r} \left\{ \sum_{j=0}^{m_k - 1} |\xi^{(j)}(\lambda_k)|^2 \right\}^{1/2}, \qquad (5)$$

is a complete Hilbert space. For each $a > 0$ the L-Fourier transform $f \to \omega f$ is a one-to-one and continuous mapping of the Hilbert space $\mathfrak{H}(a)$ onto the Hilbert space $\mathfrak{G}(a)$. In particular, the two-sided estimate holds

$$\frac{1}{d(a)} \|f\| \leq |\omega f| \leq d(a) \|f\|, \qquad f \in \mathfrak{H}(a), \qquad (6)$$

where $d(a) > 0$ and $\| \cdot \|$ is the norm of $L^2(R^+)$.

The proof of Theorem 2 is based on S. N. Bernstein' s inequality for entire functions of finite degree that are bounded on the real axis. We note that this inequality had earlier been applied to problems of spectral theory by V. A. Marchenko (see ([4])).

Put $\mathfrak{G}_0 = \bigcup \mathfrak{G}(a)$, $0 < a < \infty$, and we shall say that a sequence $\{\xi_n\} \subset \mathfrak{G}_0$ converges if there exists an $a > 0$ such that $\{\xi_n\} \subset \mathfrak{G}(a)$ and this sequence converges in the norm (5). By Theorem 2 there exists an operator $(\omega')^{-1}$, inverse to the operator ω', adjoint to the L-Fourier transform $\omega : \mathfrak{H}_0 \to \mathfrak{G}_0$. The operator $(\omega')^{-1}$ is a one-to-one mapping of $\mathfrak{H}'_0$ onto $\mathfrak{G}'_0$.

Definition. By the L-Fourier transform of an arbitrary function $f^* \in L^2_*(R^+)$ we shall mean the functional

$$\Omega f^* = (\omega')^{-1} \mathcal{P} f^*. \tag{7}$$

Corollary. In order that the L-Fourier transform of a function $f^* \in L^2_*(R^+)$ be equal to zero, it is necessary and sufficient that f^* admit the representation (4).

Denote by $\mathfrak{G}^*$ the Hilbert space of elements ξ^*, each of which is determined by a function $\xi^*(\lambda)$, $\lambda > 0$, and by the numbers $\xi^{*(j)}(\lambda_k)$, $j = 0, \dots, m_k - 1$; $k = 1, \dots, r$, with the norm in the space $\mathfrak{G}^*$ given by the formula

$$|\xi^*| = \left\{ \int_{-\infty}^{\infty} \left| \frac{\xi^*(s^2)s}{|s|+1} \right|^2 ds \right\}^{1/2} + \sum_{k=1}^{r} \left\{ \sum_{j=0}^{m_k-1} |\xi^{*(j)}(\lambda_k)|^2 \right\}^{1/2}. \tag{8}$$

It turns out that $\omega f \in \mathfrak{G}^*$ for every function $f \in L^2(R^+)^*$. To each

* If the operator L has spectral singularities, then $\mathfrak{G}^*$ is broader than the set of L-Fourier transforms of functions from $L^2(R^+)$.

to the element $\xi^* \in \mathfrak{G}^*$ we assign the functional $\Pi\xi^*$, defined by the formula

$$\langle \xi, \Pi\xi^* \rangle = \frac{1}{\pi} \int_0^{\infty} \frac{\xi(\lambda)\xi^*(\lambda)\sqrt{\lambda}}{A(\sqrt{\lambda})A(-\sqrt{\lambda})} d\lambda + \sum_{k=1}^{r} \left\{ \left(\frac{d}{d\lambda} \right)^{m_k-1} M_k(\lambda)\xi(\lambda)\xi^*(\lambda) \right\}_{\lambda=\lambda_k}, \quad \xi \in \mathfrak{G}_0. \tag{9}$$

It is not difficult to see that $\Pi\xi^* \in \mathfrak{G}'_0$ for all $\xi^* \in \mathfrak{G}^*$. Now Parseval' s equality (3) can be rewritten in the form

$$\langle f, \mathcal{P}g \rangle = \langle \omega f, \Pi\omega g \rangle, \quad f \in \mathfrak{H}_0, \quad g \in L^2(R^+), \tag{10}$$

whence, for $g \in L^2(R^+) \subset L^2_*(R^+)$,

$$\Omega g = (\omega')^{-1} \mathcal{P} g = \Pi\omega g, \tag{11}$$

so that, *up to the embedding* $\Pi : \mathfrak{G}^* \to \mathfrak{G}_0'$, the operator Ω is an extension of the operator ω.

The constructed extension Ω of the L-Fourier transform ω is an extension by continuity in the following sense. For each functional $\xi' \in \mathfrak{G}_0'$ put

$$N_a(\xi') = \sup |\langle \xi, \xi' \rangle| / |\xi|, \quad \xi \in \mathfrak{G}(a). \tag{12}$$

Let $f^* \in L^2_*(R^+)$, $f_n \in L^2(R^+)$, and for all $a > 0$

$$\lim_{n\to\infty} \int_0^a |f_n(x) - f^*(x)|^2 \, dx = 0. \tag{13}$$

Then for all $a > 0$

$$\lim_{n\to\infty} N_a(\Omega f_n - \Omega f^*) = 0. \tag{14}$$

Let us explain pictorially, although not quite rigorously, why the L-Fourier transform of the principal functions of the spectral singularities is equal to zero. If $\tilde\lambda$ is a spectral singularity, then $\omega(x, \tilde\lambda) = c\, y_1(0, \sqrt{\tilde\lambda})$, where c is some number. Let $\operatorname{Im} s > 0$ and $s \to \sqrt{\tilde\lambda}$. Then $y_1(\cdot, s) \in L^2(R^+)$ and $c\, y_1(x, s) \to \omega(x, \tilde\lambda)$ uniformly on each finite interval of variation of x. It is natural to regard the L-Fourier transform of the principal function $\omega(x, \tilde\lambda)$ as the limit of the L-Fourier transform of the function $c\, y_1(x, s)$ as $\operatorname{Im} s > 0$, $s \to (\tilde\lambda)^{1/2}$. Applying Green' s formula, we obtain

$$\int_0^\infty y_1(x, s)\omega(x, \lambda)\, dx = A(s)(s^2 - \lambda)^{-1}, \quad \operatorname{Im} s > 0, \quad \lambda > 0. \tag{15}$$

Taking into account that $A(\sqrt{\tilde\lambda}) = 0$, we see that the limit under consideration is equal to zero for $\lambda \neq \tilde\lambda$ and is equal to $A'(\sqrt{\tilde\lambda})/2\sqrt{\tilde\lambda}$ for $\lambda = \tilde\lambda$. Since $\tilde\lambda$ is a point of the continuous spectrum of the operator L, the spectral measure of the set consisting of this one point is equal to zero (this cannot be asserted when $\tilde\lambda$ belongs to the point spectrum). Consequently, the L-Fourier transform of $\omega(x, \tilde\lambda)$ is a function equal to zero almost everywhere with respect to the spectral function of the operator L. We note that an analogous argument, carried out for the case when $\tilde\lambda$ is a point of the continuous spectrum which is not a spectral singularity, shows that the L-Fourier transform of $\omega(x, \tilde\lambda)$ is equal to the δ-function.

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Note: Figure translations are in progress. See original paper for figures.