MATHEMATICS

1964

SovietRxiv

Abstract

Full Text

MATHEMATICS

M. Ya. Azbel'

ON THE SPECTRUM OF DIFFERENCE EQUATIONS WITH PERIODIC COEFFICIENTS

(Presented by Academician I. G. Petrovskii on 4 VI 1964)

1. We shall be interested in the spectrum and eigenfunctions of equations of the type*

$$\hat{L}y(x) \equiv \sum_{n=-l}^{l} f_n(x)y(x + 2\pi n\beta) = \lambda y(x); \tag{1}$$

$$f_n(x + 2\pi) = f_n(x); \qquad f_n(x) = \overline{f_n(x - 2\pi n\beta)},$$

where, without loss of generality, one may assume that $0 \leqslant \beta < 1$.

It turns out that the character of the spectrum is determined by the commensurability of the least common period 2π of the functions $f_n(x)$ with the length of the "step" $2\pi\beta$, i.e., it is closely connected with the expansion of β into a continued fraction, and has singularities at all rational points.

Since the Hermitian operator $\hat{L}$ and the unitary operator $\hat{B} = \exp(2\pi\, d/dx)$ commute, the eigenfunctions of equation (1) may be considered in advance to be subject to the condition

$$y(x + 2\pi) = e^{i\omega}y(x). \tag{2}$$

(Since the eigenfunctions of $\hat{L}$, defined in the space L_2, are generalized functions, this argument has a formally algebraic character. It can be rigorously justified for rational β.) Already from (2) it is clear that it suffices to investigate $0 \leqslant \beta < 1$.

Let us write the general solution $Y(x)$ of equation (1). Since equation (1) may be regarded as a recurrence relation, the general solution will be determined if it is prescribed (arbitrarily) on an interval of length $2\pi\beta l$. Suppose $2l$ basic solutions $(z_1(x), z_2(x), \dots, z_{2l}(x)) \equiv z(x)$ are known, for which $\operatorname{Det} \hat{\varphi} \neq 0$ $(\varphi_{ik}(x) = z_i(x + 2\pi\beta k);\ i, k = 1, 2, \dots, 2l)$ on an interval in x of length $4\pi\beta l$ (and hence on the whole axis). Then $Y(x)$ is constructed with the aid of $2l$ arbitrary periodic functions of period $2\pi\beta$, $(C_1(x), C_2(x), \dots, C_{2l}(x)) \equiv C(x)$:

$$Y(x) = C_i(x)z_i(x) = C(x)z(x), \qquad C(x + 2\pi\beta) = C(x). \tag{3}$$

(That (3) is a solution of (1) is obvious; on the other hand, $C(x)$ on an interval of length $2\pi\beta$ will be uniquely determined by the initial condition $C(x)z(x + 2\pi\beta k) = \varphi(x + 2\pi\beta k)$, $0 \leqslant k \leqslant 2l - 1$, k an integer; $\varphi(x)$ a function prescribed on an interval of length $4\pi\beta l$.)

---

* Physically, this equation corresponds to the Schrödinger equation for a charged quasiparticle with a periodic dispersion law in a constant magnetic field; this also determines the physical meaning of the solution obtained (see (1)). For the correspondence between this equation and the Schrödinger equation in a constant magnetic field for an electron in a periodic lattice, see (2).

Consequently, $z(x + 2\pi)$, which is also a solution of (1), can, in accordance with (3), be written in the form

$$z(x + 2\pi) = \hat{g}(x)z(x), \qquad \hat{g}(x + 2\pi\beta) = \hat{g}(x). \tag{4}$$

The matrix $\hat{g}$ depends on β as on a parameter; the rank of $\hat{g}$ is equal to $2l$. From (2)–(4) we find

$$C(x + 2\pi)\hat{g}(x) = e^{i\omega}C(x).$$

Taking into account the periodicity of $C(x)$ and $\hat{g}(x)$, we obtain from (5)

$$\mathbf{D}^{(1)}(x_1 + 2\pi\beta_1)\hat{h}^{(1)}(x_1) = e^{i\omega}\mathbf{D}^{(1)}(x_1); \qquad \hat{h}^{(1)}(x_1 + 2\pi) = \hat{h}^{(1)}(x_1); \tag{6}$$

$$\mathbf{D}^{(1)}(x_1 + 2\pi) = \mathbf{D}^{(1)}(x_1);$$

$$x_1 = \beta^{-1}x, \qquad \mathbf{D}^{(1)}(x_1) = C(x), \qquad \hat{h}(x_1) = \hat{g}(x), \qquad \beta_1 = \{\beta^{-1}\}. \tag{7}$$

From equations (6) it is easy to obtain for the function $D_i^{(1)}(x_1)$ a difference equation of order $2l$, analogous to equation (1), but now only a solution periodic in x_1 is sought, and the role of β is played by β_1, the fractional part of β^{-1}. The equation for $D_i^{(1)}(x_1)$, in turn, can in the same way be reduced to an equation for $D^{(2)}(x_2)$ with "step" $2\pi\beta_2 = 2\pi\{\beta_1^{-1}\}$, and so on, in accordance with the expansion of β into a continued fraction with numerators equal to unity and denominators $s_1, s_2, \dots$.

In the special case when β is a rational number ($\beta = 2\pi r/s;\ r, s$—integers), at some stage one obtains $\beta_m = 0$. Then the solvability condition for the equation for $\mathbf{D}^{(m)}(x_m)$ will determine those points x_m^0 at which $\mathbf{D}^{(m)}(x_m) = 0$, and the eigenvalues λ of equation (1),

$$\lambda = \lambda(\omega, x^0). \tag{8}$$

The eigenfunctions corresponding to the given ω and x^0 are

$$y(x) = e^{i\omega x} T(x - x^0) U(x), \qquad U(x + 2\pi r) = U(x),$$

$$T(x) = \sum_{n=-\infty}^{\infty} \delta(x + 2\pi\beta n), \tag{9}$$

($\delta(x)$ is the δ-function) and are common eigenfunctions of the operators $\hat{L}, \hat{B}^r$ and $A = \exp(i\beta^{-1}x)$.

Determining ω from (8) and requiring ω to be real, we obtain "allowed" intervals of x^0, and hence, according to (9), intervals of x in which, for the given λ, an eigenfunction may be nonzero.* (It can be obtained, for example, by integrating (9) with an arbitrary weight $G(x^0)$ over the "allowed" x^0 in an interval of length $2\pi\beta$.)

2. Let us carry out a concrete calculation of the eigenfunctions and eigenvalues for $\beta, \beta_1, \beta_2, \ldots \ll 1$, i.e. for $s_1, s_2, s_3, \ldots \gg 1$ (obviously, the case $1 - \beta_i \ll 1$ is analogous to the case $\beta_i \ll 1$) and for analytic $f_n(x)$. Since, apparently, no $s_i \sim 1$ are special points, it is natural to think that the general character of the spectrum is preserved for arbitrary β.

The principal quantity required for finding the spectrum is the matrix $\hat{g}$. There are several essentially different cases; they are conveniently classified by the form of the curves (real, owing to the Hermiticity of $\hat{L}$)

$$\varphi(x, p) = \left(\sum_{n=-l}^{l} f_n(x) \exp(inp) \right)_{\beta=0} = \lambda. \tag{10}$$

---

[*] Thus, for rational β, the eigenfunctions are singular, being, generally speaking, identically equal to zero on entire intervals of x. V. I. Matsaev observed that for irrational β poorly approximated by rational numbers, the eigenfunctions for analytic $f_n(x)$ are, apparently, analytic functions.

If these curves have no symmetries with respect to rotations in the plane (x, p), then, with accuracy exponential in $(-\beta^{-1})$, the matrix $\hat{g}$ does not depend on x, the spectrum is stable with respect to β, and is found, according to (6), from the equation $\mathrm{Det}(\hat{g} - e^{i\omega}\hat{E}) = 0$; proceeding from other considerations, this spectrum was obtained in the papers ($^{2-6}$).

We shall demonstrate the nature of the spectrum in the presence of symmetry in the case of an axis of symmetry of fourth order, passing perpendicularly to the plane (x, p) through the point $x = p = 0$. We shall assume that $(\partial^2\varphi/\partial x^2)_{x=0,\ p=\pi} \neq 0$. In this case the matrix $\hat{g}$ can be found by matching the solution in the form WKB: $Y(x) = \exp(\beta^{-1}\sigma_0(x) + \sigma_1(x) + \beta\sigma_2(x) + ...)$, valid far from the points $x = q\pi$, q an integer, with the solution in neighborhoods of these points. The latter is easy to obtain* if one notes that near $x = (2q+1)\pi$ there exist such $z(x)$ that, up to higher-order small terms in β, $z(x + \iota\beta) = z(x) + \iota\beta z'(x) + \frac{1}{2}\iota^2\pi^2\beta^2 z''(x)$; near $x = 2q\pi$ a similar expansion is given by $z(x)\exp(-\frac{1}{2}i\beta^{-1}x)$.

In the principal approximation in $\hat{\beta}$, $g(x)$ has the form:

$$\hat{g}(x) = \begin{pmatrix} \Delta^{-1}t^2e^{-i\Sigma} - A^{-1} & -\Delta^{-1}ite^{i\Sigma} + itA \\ \Delta^{-1}ite^{-i\Sigma} - itA^{-1} & \Delta^{-1}t^2e^{i\Sigma} - A \end{pmatrix}; \qquad A = e^{i\beta^{-1}x};$$

$$\Sigma = (4\pi\beta)^{-1}S(\lambda); \qquad \Delta = \exp(-\pi\rho); \qquad t = \sqrt{1 + \Delta^{-2}e^{i\chi}}; \qquad \lambda_0 = \varphi(0, \pi); \tag{11}$$

$$\chi = \rho\ln(\rho e^{-1}) - \frac{1}{2i}\ln\frac{\Gamma(1/2 + i\rho)}{\Gamma(1/2 - i\rho)}, \qquad \chi(\infty) = 0,$$

$$\rho = \left|\pi\beta\,\frac{\partial^2\varphi}{\partial x^2}\right|^{-1}_{x=0,\ p=\pi} |\lambda - \lambda_0|,$$

where $S(\lambda)$ is the area bounded by the curve (10).

The rank of the matrix turned out to be equal to two because, for $\beta \ll 1$, near the point $x = 0$, with accuracy exponential in $(-\beta^{-1})$, only two branches of the solution "get intertwined."

For $\beta_1 \ll 1$, equation (6) is substantially simplified and can be written in the form:

$$\frac{1}{2}\left(e^{-i\omega}D_1^{(1)}(x_1 + 2\pi\beta_1) + e^{i\omega}D_1^{(1)}(x_1 - 2\pi\beta_1)\right) + \cos x_1 \cdot D_1^{(1)}(x_1) = \lambda^{(1)}D_1^{(1)}(x_1); \tag{12}$$

$$D_1^{(1)}(x_1 + 2\pi) = D_1^{(1)}(x_1).$$

The eigenvalues λ, for a given $\lambda^{(1)}$, depend on the integer number m and are determined from the equation

$$\Sigma(\lambda) = \left(m + \frac{1}{2}\right)\pi + 2\chi(\lambda) - (-1)^m \arcsin\left(\Delta(\lambda)\left(1 + \Delta^2(\lambda)\right)^{-1}\lambda^{(1)}\right). \quad (12a)$$

For almost all levels $m \gg 1$ (since $\Sigma \sim S(\lambda)\beta^{-1}$).

Thus the equation for $\lambda^{(1)}$ proves, for $\beta, \beta_1 \ll 1$, to be universal with respect to the form of the operator $\hat{L}$. It is clear that the smallness of β_1 makes it possible at once, by the ready-made formula, to write down the equation with β_2 to which (12) is reduced, while the smallness of β_2 makes it possible to write out the formula for $\lambda^{(1)}$, and so on. Since, in these approximations, $\lambda^{(n+1)}$ and $D^{(n+1)}(x_{n+1})$ depend only on β_{n+1}, they are, together with β_{n+1}, periodic functions of β_n^{-1} with period equal to unity, and have an essential singularity at $\beta_n = 0$. From what has been said it is also clear how the deformation of the spectrum occurs under a continuous change of β.

---

* Near these points one can also find the general solution $Y(x)$, expanding $f_n(x)$ in a series, retaining in it the first non-vanishing terms and passing to the equation for the image in the Fourier transform of the function $Y(x)$. For the latter equation a matching of a solution of type WKB with solutions near the "special" points is also possible. Such an approach makes it possible to extend the method described here for finding the spectrum (for $\beta \ll 1$) to differential-difference equations with periodic coefficients.

Thus, the structure of the spectrum has the following character. The eigenvalues consist of $s_1 = [\beta^{-1}]$ quasi-equidistant levels (the distance between them is of order β), each of which is split in a universal way into $s_2 = [\beta_1^{-1}]$ sublevels, the distance between which is of order $\beta\beta_1\Delta(m)$ and decreases exponentially (identically for all sublevels) as $|\lambda(m) - \lambda_0|$ increases. Each of the sublevels is similarly split into sub-sublevels, etc. (This process terminates only in the special case of rational β, when bands are obtained at the last stage (see (8)). The levels of each "superstructure" oscillate according to their "own" β_i^{-1} with period 1, and every rational point β is a singular point of the spectrum (this can be proved in the general case, without the assumption of smallness of all preceding β_k, $k < i$).

Received
1 IV 1964

REFERENCES

[1] M. Ya. Azbel, ZhETF, **46**, 109 (1964).
[2] G. E. Zilberman, ZhETF, **32**, 296 (1957); **34**, 515 (1958).

[3] I. M. Lifshits, A. M. Kosevich, ZhETF, **29**, 730 (1955).
[4] L. Onsager, Phil. Mag., **43**, 1006 (1952).
[5] M. Ya. Azbel, ZhETF, **39**, 1276 (1960).
[6] G. E. Zilberman, ZhETF, **33**, 387 (1957).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.