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**Abstract**

**Full Text**

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## ON SOME PROPERTIES OF SOLUTIONS OF NONLINEAR EQUATIONS OF HYPERBOLIC TYPE IN A HILBERT SPACE

*(Presented by Academician I. N. Vekua on 20 III 1964)*

We shall say that the linear unbounded operators  $A(t)$  ( $0 \leq t < \infty$ ), acting in the real Hilbert space  $H$  and having an everywhere dense domain of definition  $D(A)$ , independent of  $t$ , satisfy condition (A) if they are self-adjoint, positive definite, if the strong derivative  $A'(t)$  exists, and

$$(A'(t)x, x) \leq \alpha(t)(A(t)x, x), \quad x \in D(A),$$

where  $\alpha(t)$  is a certain function defined on  $[0, \infty)$ .

Further, we shall say that the unbounded operators  $B[t, x]$  ( $0 \leq t < \infty$ ), acting in  $H$ , satisfy condition (B) if

$$(B[t, x], x) \geq -\frac{\alpha(t)}{2} \|x\|^2, \quad x \in D(B).$$

By  $\tilde{E}$  we denote the set of abstract functions  $x(t)$  ( $0 \leq t < \infty$ ) with values in  $H$ , differentiable with respect to  $t$  and such that, for each fixed  $t$ ,  $x(t) \in D(A^{1/2})$ . Let  $x(t) \in \tilde{E}$ , and introduce the notation

$$\|x(t)\|_E = \|\dot{x}(t)\| + \|A^{1/2}(t)x(t)\|.$$

### 1. Boundedness of solutions.

Let  $F(t, x)$  ( $0 \leq t < \infty$ ,  $x \in D(A^{1/2})$ ) be a family of nonlinear functionals differentiable in the sense of Gâteaux:

$$\lim_{\lambda \rightarrow 0} \frac{F(t, x + \lambda h) - F(t, x)}{\lambda} = (P(t, x), h).$$

Consider second-order equations with a nonlinear potential operator

$$\ddot{x} + A(t)x + P(t, x) = 0, \tag{1,1}$$

$$\ddot{y} + A(t)y + B[t, \dot{y}] + P(t, y) = 0. \quad (1,2)$$

**Theorem 1.** Let the operators  $A(t)$  and  $B[t, x]$  satisfy respectively conditions (A) and (B). Suppose that there exists  $F'_t(t, x)$ , and

$$F'_t(t, x) \leq \alpha(t)E(t, x) \quad (0 \leq t < \infty, x \in D(A^{1/2})).$$

Then, for any solution  $x(t)$  of equation (1,1) and  $y(t)$  of equation (1,2) ( $x(0), y(0) \in D(A^{1/2}(0))$ ), for  $t \geq 0$  the estimates hold

$$\begin{aligned} \|\dot{x}\|^2 + \|A^{1/2}(t)x\|^2 + 2F(t, x) &\leq \{\|x(0)\|^2 + \\ &+ \|A^{1/2}(0)x(0)\|^2 + 2F[0, x(0)]\} \exp \left[ \int_0^t |\alpha(s)| ds \right], \end{aligned} \quad (1,3)$$

$$\|A^{1/2}(t)x\|^2 + 2F(t, x) \leq \{\|A^{1/2}(0)x(0)\|^2 + 2F[0, x(0)]\} \exp \left[ \int_0^t \alpha(s) ds \right], \quad (1,4)$$

$$\begin{aligned} \|\dot{y}\|^2 + \|A^{1/2}(t)y\|^2 + 2F[t, y] &\leq \\ &\leq \{\|\dot{y}(0)\|^2 + \|A^{1/2}(0)y(0)\|^2 + 2F[0, y(0)]\} \exp \left[ \int_0^t \alpha(s) ds \right]. \end{aligned} \quad (1,5)$$

**1.1.** We present some consequences obtained from Theorem 1. Let  $F(t, x) \geq 0$ . Then from (1,3)–(1,5) we have

$$\|x(t)\|_E \leq \{\|x(0)\|_E + \sqrt{2F[0, x(0)]}\} \exp \left[ 2 \int_0^t |\alpha(s)| ds \right], \quad (1,3')$$

$$\|A^{1/2}(t)x(t)\| \leq \{\|A^{1/2}(0)x(0)\| + \sqrt{2F[0, x(0)]}\} \exp \left[ 2 \int_0^t \alpha(s) ds \right], \quad (1,4')$$

$$\|y(t)\|_E \leq \{\|y(0)\|_E + \sqrt{2F[0, y(0)]}\} \exp \left[ 2 \int_0^t \alpha(s) ds \right] \quad (1,5')$$

for  $t \geq 0$ . From (1,3'), (1,5') it follows that if

$$\int_0^{\infty} |\alpha(t)| dt < \infty, \quad (1,6)$$

then  $\|x(t)\|_E$  and  $\|y(t)\|_E$  are bounded, and the zero solutions of equations (1,1) and (1,2) are stable. A similar question from a somewhat different point of view for the linear problem was considered in (1). If we assume that

$$\int_0^{\infty} \alpha(t) dt = -\infty, \quad (1,7)$$

then from (1,4') and (1,5') it follows that  $\lim_{t \rightarrow \infty} \|A^{1/2}(t)x\| = 0$ ,  $\lim_{t \rightarrow \infty} \|y(t)\|_E = 0$ , i.e., the zero solutions of equations (1,1) and (1,2) are asymptotically stable.

1,2. Let  $H = L^2(G)$ , where  $G$  is a bounded domain of  $n$ -dimensional Euclidean space with smooth boundary  $\Gamma$ , and

$$A(t)x \equiv - \sum_{i,j=1}^n \frac{\partial [a_{ij}(t,s)x_{s_j}]}{\partial s_i} + ax, \quad x|_{\Gamma} = 0, \quad (1,8)$$

where  $a_{ij} = a_{ji}$ ,  $a > 0$ ,  $\sum a_{ij}\xi_i\xi_j \geq 0$ . We shall assume that  $a_{ij}$ ,  $a$ ,  $\partial a_{ij}/\partial s_i$  have derivatives with respect to  $t$  and are bounded above by the function  $\alpha(t)$  on  $[0, \infty)$ . Then the operator-function  $A(t)$ , defined by formula (1,8), on the set  $W_2^0(G)$  satisfies condition (A) (see (2)). An example of an operator satisfying condition (B) is the operator

$$B[t, x] \equiv - \sum_{i,j=1}^n \frac{\partial [b_{ij}(t,s)x_{s_j}]}{\partial s_j} + g(t, x), \quad x|_{\Gamma} = 0,$$

if we assume that

$$\sum_{i,j=1}^n b_{ij}\xi_i\xi_j \geq -\frac{\alpha(t)}{4} \sum_{i=1}^n \xi_i^2, \quad g(t, \dot{x}) \dot{x} \geq -\frac{\alpha(t)}{4} \dot{x}^2.$$

Let  $\Phi(t, x)$  be a continuous scalar function, positive for  $x > 0$  and  $t \in [0, \infty)$ ; moreover,

$$\dot{\Phi}_t(t, x) \leq \alpha(t)\Phi(t, x), \quad |\Phi_x(t, x)| \leq Mx^m,$$

where  $m = \frac{1}{n-2}$  if  $n > 2$ ;  $m < \infty$  if  $n \leq 2$ . Then the functional

$$F(t, x) \equiv \frac{1}{2} \int_G \Phi[t, x^2(t, s)] ds \geq 0$$

satisfies the conditions of theorem 1, and  $P(t, x) = \Phi'_x(t, x^2)x$ .

1.3. If we set  $H = E^n$ , then, as a special case, from theorem 1 one can obtain boundedness and stability of solutions of systems of ordinary differential equations, which is a generalization of the results of (3).

## 2. On some other properties of solutions

We shall now consider the equation

$$\ddot{x} + A(t)x = f(t, x) \tag{2.1}$$

in the case when the nonlinear operator  $f(t, x)$  ( $0 \leq t < \infty$ ,  $x \in S_r : \|A^{1/2}x\| \leq r$ ) is not necessarily potential.

2.1. By  $x(t)$  and  $y(t)$  ( $0 \leq t < \infty$ ,  $x, y \in S_r$ ) we denote, respectively, solutions of equation (2.1) satisfying the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \tag{2.2}$$

$$y(0) = y_0, \quad \dot{y}(0) = \dot{y}_0. \tag{2.3}$$

**Theorem 2.** *Let the operator  $A(t)$  satisfy condition (A). Suppose that for  $t \in [0, \infty)$ ,  $x, y \in S_r$  one has*

$$\|f(t, x) - f(t, y)\| \leq K(t, r)\|A^{1/2}(t)(x - y)\|, \tag{2.4}$$

where  $\int_0^\infty K(t, r) dt < \infty$ . Finally, suppose that  $x_0 - y_0 \in D(A^{1/2}(0))$ .

Then the inequality holds

$$\|x(t) - y(t)\|_E \leq c\|x(0) - y(0)\| \exp \left[ \int_0^t |\alpha(s)| ds \right]. \tag{2.5}$$

If we put  $x_0 = y_0$ ,  $\dot{x}_0 = \dot{y}_0$  and assume that (1.6) is fulfilled, then from estimate (2.5) it follows that the solution of problem (2.1)–(2.2) is unique. It also follows from estimate (2.5) that, if (1.6) is fulfilled, then the solution of problem (2.1)–(2.2) is stable.

Let us note that analogous facts are also valid for solutions of the equation

$$\ddot{y} + A(t)y + B[t, \dot{y}] = f(t, y), \quad (2.6)$$

if  $B[t, y]$  satisfies condition (B). Moreover, if condition (1.6) is replaced by condition (1.7), then the solution of equation (2.6) is also asymptotically stable.

2.2. Let  $y(t)$  ( $0 \leq t < \infty$ ,  $y \in S_r$ ) be a twice differentiable solution of the problem

$$A(t)y + B[t, \dot{y}] = f(t, y), \quad y(0) = x_0, \quad \dot{y}(0) = \dot{x}_0. \quad (2.7)$$

Then  $y(t)$  is simultaneously a solution of the problem

$$\ddot{y} + A(t)y + B[t, \dot{y}] = f(t, y) + \ddot{y}, \quad y(0) = x_0, \quad \dot{y}(0) = \dot{x}_0. \quad (2.8)$$

We shall say that the solutions of problem (2.6)–(2.2) are **stabilizable to the solution of problem (2.8)** if

$$\lim_{t \rightarrow \infty} \|x(t) - y(t)\|_E = 0.$$

**Theorem 4.** *Let the operator  $A(t)$  satisfy condition (A), and let the operator  $B[t, \dot{x}]$  satisfy the condition*

$$(B[t, \dot{x}], \dot{x}) \geq \frac{1 - \alpha(t)}{2} \|\dot{x}\|^2.$$

*Suppose that the operator  $f(t, x)$  satisfies condition (2.4). Finally, suppose that (1.7) is fulfilled. Then the solutions of problem (2.6)–(2.2) are stabilizable to the solution of problem (2.8).*

**3. Existence of a solution.** We now consider the question of existence of a solution to problem (2.1)–(2.2).

We shall consider the case when  $A(t) = A$  and  $t \in [0, T]$ .

3.1. Define a sequence of functions  $x_n(t)$  ( $n = 1, 2, \dots$ ,  $t \in [0, T_1]$ , where  $T_1 \leq T$ ) by the equalities (see (4)):

$$x_n(t) = x_0 \left( -\frac{T_1}{n} \leq t \leq 0 \right), \quad \dot{x}_n(0) = \dot{x}_0,$$

$$\ddot{x}_n(t) + Ax_n(t) = f \left[ t - \frac{T_1}{n}, x_n \left( t - \frac{T_1}{n} \right) \right] \quad (0 \leq t \leq T_1). \quad (3.1)$$

Suppose that

$$\|x(0)\|_E < \frac{r}{2} \exp \left[ - \int_0^{T_1} [\alpha(t) + K(t)] dt \right], \quad (3.2)$$

$$\int_0^{T_1} \|f(t, 0)\|^2 dt < \infty. \quad (3.3)$$

Then it is obvious that  $T_1$  can be chosen so that  $x_n(t) \in S_r$ .

If the function  $x^*(t)$  is a solution of the equation

$$x(t) = \cos(tA^{1/2})x_0 + \sin(tA^{1/2})(A^{-1/2}x_0) + \int_0^t \sin[(t-s)A^{1/2}](A^{-1/2}f[s, x(s)]) ds \quad (3.4)$$

and  $x^*(t) \in \widetilde{E}$ , then we shall call it a **generalized solution** of problem (2.1)–(2.2).

**Theorem 4.** Let the operator  $A$  be self-adjoint, positive definite, and have an everywhere dense domain of definition, and suppose condition (3.2) is satisfied. Let the operator  $f(t, x)$  satisfy condition (2.4), where

$$\int_0^{T_1} K^2(t, r) dt < \infty$$

and suppose (3.3) is satisfied. Then there exists a (unique!) generalized solution  $x^*(t)$  ( $t \in [0, T_1]$ ) of problem (2.1)–(2.2), and it is the limit in the norm of  $\widetilde{E}$  of the sequence  $\{x_n(t)\}$  defined by equalities (3.1).

**Theorem 5.** Let the operator  $A$  satisfy the condition of Theorem 5 and suppose (3.2) is satisfied. Let  $f(t, x)$  be a compact and bounded operator defined on  $[0, T] \times S_r$ . Then there exists at least one generalized solution of problem (2.1)–(2.2), defined on the segment  $[0, T_1]$ .

We have presented here theorems giving local existence of a solution of problem (2.1)–(2.2). Using Theorem 1, one can give a nonlocal existence theorem.

**Theorem 6.** Let the operator  $A$  satisfy the conditions of Theorem 5. Let condition (3.2) be satisfied. Finally, let the functional  $F(t, x)$  satisfy the conditions of Theorem 1. Then, if the operator  $P(t, x)$  satisfies those conditions of Theorem 5 (Theorem 6) which  $f(t, x)$  satisfies, there exists a unique (at least one) generalized solution of problem (1.1), defined on the whole segment.

We note that in the work <sup>5</sup> an existence theorem for a solution close to Theorem 7 was obtained in connection with the study of the scattering operator for hyperbolic equations.

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*Note: Figure translations are in progress. See original paper for figures.*

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