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Abstract

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MATHEMATICS

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ON A PRIORI ESTIMATES OF SOLUTIONS OF DEGENERATING ELLIPTIC EQUATIONS

(Presented by Academician A. N. Kolmogorov, 17 IV 1964)

Let in the n -dimensional space R^n there be given an elliptic, possibly degenerating, differential operator

$$L = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x^i}, \quad (1)$$

and a bounded domain D with boundary Γ . Consider the Dirichlet problem

$$\begin{aligned} Lu(x) - c(x)u(x) &= 0, \quad x \in D, \\ u(x)|_{\Gamma} &= \psi(x), \end{aligned} \quad (2)$$

where $c(x)$ is a continuous nonnegative function in R^n ; $\psi(x)$ is a continuous function defined on Γ . We assume that the coefficients of the operator $L - c(x)$ are bounded and satisfy the Lipschitz condition in the whole space. In addition, we assume that there exists a matrix $\sigma(x) = \{\sigma_{ij}(x)\}$ with coefficients satisfying the Lipschitz condition such that $\sigma^2(x) = \{a_{ij}(x)\}$. Denote by K the Lipschitz constant (the same one for the functions $\sigma_{ij}(x), b_i(x)$).

In the work ⁽²⁾ a generalized solution of problem (2) was constructed, and conditions ensuring the continuity and uniqueness of this solution were given. The purpose of the present note is to establish certain a priori estimates which make it possible to judge the smoothness properties of the generalized solution. A priori estimates of the derivatives are valid under certain restrictions. The example given below shows their essential nature. Under very weak restrictions on the coefficients, an a priori estimate of solutions in the Hölder norm holds.

Consider the stochastic equation

$$x_t^x = \int_0^t \sigma(x_u) d\xi_u + \int_0^t b(x_u) du. \quad (3)$$

Here $\xi_u = \{\xi_u^1, \dots, \xi_u^n\}$ is an n -dimensional Wiener process, $b(x) = \{b_1(x), \dots, b_n(x)\}$. It is known that a solution of this equation exists (see ⁽¹⁾). From it one can construct a Markov process $X = \{x_t(\omega), P_x\}$. With the aid of this process, the solution of problem (2) is written in the form:

$$u(x) = M_x \psi(x_\tau) \exp \left\{ - \int_0^\tau c(x_u) du \right\},$$

where $\tau = \inf\{t : x_t \in \Gamma\}$.

A point $x_0 \in \Gamma$ will be called **normally regular** if

$$\frac{M_x \tau}{\rho(x, \Gamma)} < A < \infty$$

as $x \rightarrow x_0$ ($\rho(x, \Gamma)$ is the distance from x to the set Γ). Sufficient conditions for normal regularity can be given in terms of the coefficients. (It is sufficient, for example, that at the point x_0 the coefficient be nonzero ...)

diffusion in the direction of the normal to Γ , or that the vector $b(x)$ be directed outward from the domain D .) If the constant A can be chosen the same for all points $x_0 \in \hat{\Gamma} \subset \Gamma$, then the points of the set $\hat{\Gamma}$ will be called **uniformly normally regular**.

As shown in (2), in order that the generalized solution be unique it is necessary (for $c(x) \equiv 0$) and sufficient that $P_x\{\tau < \infty\} = 1$ for all $x \in D$. We shall say that the **trajectories leave the boundary uniformly rapidly** if

$$\lim_{T \rightarrow \infty} P_x\{\tau > T\} = 0$$

uniformly in $x \in D$. In order that the trajectories leave the boundary uniformly rapidly, it is sufficient that at least one coefficient of the operator L be different from zero in the closed domain $D \cup \Gamma$ (see (2)).

Lemma 1. *If the trajectories of the process leave the boundary uniformly rapidly, then for some $\alpha > 0$*

$$P_x\{\tau > t\} \leq e^{-\alpha t}.$$

Remark. One can give an estimate for α in terms of the coefficients of the operator and the diameter d of the domain D . For example, if $a_{ii}(x) > a > 0$ for $x \in D$ and $b_i(x) \equiv 0$, then $\alpha > a/d^2$. An estimate of the constant α in terms of the coefficients can also be given in a more general case.

Theorem 1. *Suppose that the boundary points are uniformly normally regular and that the boundary function satisfies the Hölder condition:*

$$|\psi(x) - \psi(y)| < K_1|x - y|^{\gamma_1}.$$

Suppose, moreover, that the trajectories of the process X leave the boundary uniformly rapidly. Then, for

$$\gamma < \min \left(\gamma_1, \frac{\alpha}{2n^2(k^2 + k)} \right) = \bar{\gamma},$$

the a priori estimate holds

$$|u(x) - u(y)| < N_2|x - y|^\gamma.$$

The constant N_1 depends on the constants K , K_1 , α and on the maximum of the moduli of the coefficients.

From the example given at the end of the note it follows that, without restrictions of the type of the inequalities (see below), one cannot obtain an a priori estimate for arbitrary γ . If $\min c(x) = c > 0$, then in the expression for $\bar{\gamma}$ one may replace α by $\alpha + c$.

Theorem 2. *Suppose that the conditions of Theorem 1 are fulfilled with $\gamma_1 = 1$, and let*

$$\alpha + \min_{x \in D} c(x) > 2n^2(k^2 + k).$$

Then the a priori estimate is valid

$$|u(x) - u(y)| < N_2|x - y|.$$

Remark. If a part $\hat{\Gamma}$ of the boundary Γ consists of unattainable points (see (2)), then in Theorems 1 and 2 it is sufficient to require uniform normal regularity on the set $\Gamma \setminus \hat{\Gamma}$.

Theorem 3. *Suppose that the operator L does not degenerate in a neighborhood of the boundary and that the trajectories of the process X leave the boundary of the domain D uniformly rapidly. Then there exists a function $f(n, r, K)$ such that, if*

$$f(n, r, K) < \alpha + \min_{x \in D} c(x), \tag{4}$$

then the a priori estimate holds

$$\max_{\substack{x, i_1, \dots, i_n \\ i_1 + i_2 + \dots + i_n = r}} \left| \frac{\partial^r u(x)}{\partial x_1^{i_1} \partial x_2^{i_2} \dots \partial x_n^{i_n}} \right| < N_3.$$

The constant N_3 depends on the first r derivatives of the coefficients, the width of the degeneracy strip, the ellipticity constant inside this strip, and also on $\max_{x \in \Gamma} |\psi(x)|$ and $\beta = f(n, r, K) - \alpha - \min_D c(x)$.

The function $f(n, r, k)$ is a polynomial in K , whose coefficients depend on n and r ; it can be computed explicitly by means of recurrence relations. The example given below shows the essential nature of condition (4).

From the formulated results, the usual methods yield theorems on the smoothness of solutions of degenerating equations. If an inequality of type (4) is not assumed, then the generalized solution will only satisfy a Hölder condition.

A theorem analogous to Theorem 3 can also be proved in the case of a mixed problem for a parabolic equation. In this case, the condition of nondegeneracy at the lower boundary may be replaced by a smoothness requirement on the initial function.

Using the generalized solution of the problem with oblique derivative constructed in (4), and the formulated a priori estimates, one can study the smoothness of solutions of the problem with oblique derivative for degenerating equations.

In the special case (degeneration occurs only in the direction x^1), results analogous to Theorems 2 and 3 were obtained in (3).

The proof of the formulated assertions is carried out by purely probabilistic methods—by studying the trajectories of the corresponding Markov process.

The obtained a priori estimates make it possible to construct a generalized solution of the Dirichlet problem for degenerating quasilinear equations and to study its smoothness.

Let us give an example showing the essential nature of the conditions of Theorems 1, 2, 3. Let the domain D be the square:

$$D = \{x, y : |x| < 1, |y| < 1\}.$$

Denote by $\varphi(x, y)$ an infinitely differentiable function, even in y , equal to zero outside an ε -neighborhood of the boundary Γ of the square D . Consider the Dirichlet problem

$$Lu = \alpha \frac{\partial^2 u}{\partial x^2} + \beta y \frac{\partial u}{\partial y} + \varphi(x, y) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0,$$

$$u(x, y)|_{\Gamma} \equiv y.$$

The operator L does not degenerate in a neighborhood of the boundary, and $\alpha > 0$ throughout the whole domain D ; therefore this equation has a unique continuous generalized solution. This solution can be written in the form $u(x, y) = M_{(x, y)} y_{\tau}$, where (x_t, y_t) is the process governed by the operator L , and τ is the first time at which the trajectory reaches the boundary. Obviously, $u(0, 0) = 0$. It can be shown that

$$u(0, y) = M_{0, y} y_{\tau} > cy^{\alpha/\beta}.$$

Hence we obtain

$$u(0, y) - u(0, 0) > cy^{\alpha/\beta}.$$

It follows from the last inequality that the function $u(x, y)$, without additional assumptions on the size of α/β , may fail to have derivatives. For any γ one can choose α/β so small that the function $u(x, y)$ will not satisfy the Hölder condition with exponent γ .

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