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Abstract

Full Text

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ON CERTAIN FUNCTION SPACES

DIRECT AND INVERSE EMBEDDING THEOREMS

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We shall assume (following the notation of S. M. Nikol' skii ([3],[8])) that e is a subset of the natural numbers $e_n = \{1, \dots, n\}$. If $K = (k_1, \dots, k_n)$ is a given vector, then let $K^e = (k_1^e, \dots, k_n^e)$, where

$$k_j^e = \begin{cases} k_j, & \text{for } j \in e, \\ 0, & \text{for } j \in e_n \setminus e. \end{cases}$$

We shall denote by e_k the support of the vector K, i.e. the smallest subset $e \subset e_n$ for which $K^e = K$. Let a vector $r = (r_1, \dots, r_n)$ with nonnegative components be given. Put $r_j = \bar{r}_j + \alpha_j$, where $\bar{r}_j$ is the greatest integer less than r_j, so that $0 < \alpha_j \leq 1$, and if $r_j = 0$, then $\bar{r}_j = 0$. Thus the vector r uniquely determines the vector $\bar{r} = (\bar{r}_1, \dots, \bar{r}_n)$.

Let $f(x)$ be a smooth function defined in $E^n = \{x = (x_1, \dots, x_n)\}$. Denote by $f^{(\bar{r}_\tau, \bar{r}^e)}(x)$ the partial derivative of the function $f(x)$ of order $(\bar{r}_\tau, \bar{r}^e)$ with respect to the variables (x_τ, x^e), respectively, where e is any fixed subset of the set e_r, and $\tau \in e_r \setminus e$. The order of differentiation is arbitrary, for example:

$$f^{(\bar{r}_\tau, \bar{r}^e)}(x) = \frac{\partial^{\bar{r}_\tau}}{\partial x_\tau^{\bar{r}_\tau}} \frac{\partial^{\bar{r}_1^e}}{\partial x_1^{\bar{r}_1^e}} \cdots \frac{\partial^{\bar{r}_n^e}}{\partial x_n^{\bar{r}_n^e}} f(x).$$

For any $\tau \in e_r \setminus e$ put

$$\|f(x)\|_{L^{\bar{r}_\tau, \bar{r}^e}_{p, x_\tau, x^e}(E^n)} = \left(\int_0^\infty \frac{dt_\tau}{t_\tau^{1+p\alpha_\tau}} \prod_{j \in e} \int_0^\infty \frac{\left\| \Delta_\tau^{2, \omega^e} \Delta_{t_\tau, t^e}^{(\bar{r}_\tau, \bar{r}^e)} f(x) \right\|_{L_p(E^n)}^p}{t_j^{1+p\alpha_j}} \, dt_j \right)^{1/p},$$

where $\omega = (1, \dots, 1)$ is a vector whose components consist only of ones, $\Delta_\tau^{2, 2\omega^e} \Delta_{t_\tau, t^e} f$ is the finite difference of order 2 with respect to the variable x_τ and of order $2\omega^e$ with respect to the variables x^e, with steps t_τ and t^e, respectively; $1 \leq p \leq \infty$.

Definition 1. The space $\mathcal{L}_p^{(r)}(e; E^n)$ will mean the closure of the set of smooth finite functions in the norm

$$\|f(x)\|_{L_p^{(r)}(e;E^n)} = \sum_{\tau \in e_r \setminus e} \|f(x)\|_{L^{\bar{r}_\tau, \bar{r}^e}_{p, x_\tau, x^e}(E^n)}. \tag{1}$$

If e_0 is the empty set and $e_r = e_n$, then the space $\mathcal{L}_p^{(r)}(e_0, E^n)$ is the space $\mathcal{L}_p^{(r_1, \ldots, r_n)}(E^n)$, which was considered earlier by V. P. Il' in, V. A. Solonnikov ([6]), and others.

Definition 2. The space $B_{p_0,p}^{(r)}(e; E^n)$ will mean the closure of the set of smooth finite functions in the norm

$$\|f(x)\|_{B^{(r)}_{p_0,p}(e;E^n)} = \|f(x)\|_{L_{p_0}(E^n)} + \|f(x)\|_{L_p^{(r)}(e;E^n)}, \tag{2}$$

where $\|f(x)\|_{L_{p_0}(E^n)}$ is the norm of the function in $L_{p_0}(E^n)$ $(1 \leq p_0 \leq \infty)$.

If e_0 is the empty set and $e_r = e_n$, then $B_{p_0,p}^{(r)}(e_0; E^n)$ coincides with the well-known space $B_{p_0,p,\ldots,p}^{r_1,\ldots,r_n}(E^n)$, defined for $p_0 = p$ by O. V. Besov ([7]), the theory of which was later developed by V. P. Il' in ([5]), and others.

Definition 3. We shall say that a function $f(x)$ belongs to the space $S_p^{(r)}\mathcal{L}(E^n)$ if it belongs simultaneously to all $\mathcal{L}_p^{(r)}(e; E^n)$ for every e from e_r. We define the norm in this space as follows:

$$\|f(x)\|_{S_p^{(r)}\mathcal{L}(E^n)} = \sum_{e \subset e_r} \|f(x)\|_{L_p^{(r)}(e;E^n)}. \tag{3}$$

Definition 4. We shall say that a function $f(x)$ belongs to the space $S_{p_0,p}^{(r)}B(E^n)$ if it belongs simultaneously to all $B_{p_0,p}^{(r)}(e; E^n)$ for every e from e_r. We define the norm in this space as follows:

$$\|f(x)\|_{S^{(r)}_{p_0,p}B(E^n)} = \sum_{e \subset e_r} \|f(x)\|_{B^{(r)}_{p_0,p}(e;E^n)}. \tag{4}$$

It is obvious that the norm (4) and the following norm are equivalent:

$$\|f(x)\|^*_{S^{(r)}_{p_0,p}B(E^n)} = \|f(x)\|_{L_{p_0}(E^n)} + \|f(x)\|_{S_p^{(r)}\mathcal{L}(E^n)}. \tag{5}$$

Put $B_p^{(r)}(e; E^n) = B_{p,p}^{(r)}(e; E^n)$, $S_p^{(r)}B(E^n) = S_{p,p}^{(r)}B(E^n)$. Some embedding theorems have been obtained for the classes $B_p^{(r)}(e; E^n)$ and $S_p^{(r)}B(E^n)$, which are a development of S. M. Nikol' skii' s results for the classes $S_p^{(r)}H$ and $S_p^{(r)}W$.

Theorem 1. Let $f(x) \in S_p^{(r)}B(E^n)$, where $r_i > 0$ $(i = 1, \dots, n)$, $p > 1$. Let natural numbers ν_i $(i = 1, \dots, n)$ and m $(0 < m \leq n)$ be given, as well as numbers q $(p \leq q \leq \infty)$, ρ_s $(s = 1, \dots, m)$, and suppose that the conditions

$$\varepsilon = 1 - \frac{1}{p}\sum_{j=1}^{n}\frac{1}{r_j} - \sum_{j=1}^{n}\frac{\nu_j}{r_j} + \frac{1}{q}\sum_{j=1}^{m}\frac{1}{r_j} > 0, \tag{A}$$

$$0 < \rho_s \leq \varepsilon r_s \qquad (s = 1, \dots, m). \tag{B}$$

are satisfied.

Let e be any fixed subset of the set e_n. Then, for any fixed $x_{m+1}, \dots, x_n$, the function $f^{(\nu)}(x) \in \mathcal{L}_q^{(\rho)}(e \cap e_m; E^m)$ and the inequality

$$\|f^{(\nu)}(x)\|_{L_q^{(\rho)}(e\cap e_m;E^m)} \leq C\|f\|_{S_p^{(r)}B(E^n)} \tag{6}$$

holds, where $\nu = (\nu_1, \dots, \nu_n)$, $e_m = \{1, \dots, m\}$.

Theorem 2. Let $f(x) \in S_p^{(r)}B(E^n)$, where $r_i > 0$ $(i = 1, \dots, n)$, $p > 1$. Let natural numbers ν_i $(i = 1, \dots, n)$ and m $(0 < m \leq n)$ be given, as well as numbers q $(p \leq q \leq \infty)$, ρ_s $(s = 1, \dots, m)$, and let the conditions (A), (B) of Theorem 1 be satisfied. Then, for any fixed $x_{m+1}, \dots, x_n$, the function $f^{(\nu)}(x) \in S_q^{(\rho)}\mathcal{L}(E^m)$ and the inequality

$$\|f^{(\nu)}(x)\|_{S_q^{(\rho)}\mathcal{L}(E^m)} \leq C\|f\|_{S_p^{(r)}B(E^n)} \tag{7}$$

holds.

Theorem 3. Let $f \in S_p^{(r)}B(E^n)$, where r_i $(i = 1, \dots, n)$, $p > 1$. Let natural numbers ν_i $(i = 1, \dots, n)$ and m $(0 < m \leq n)$ be given, as well as numbers q $(p \leq q \leq \infty)$, ρ_s $(s = 1, \dots, m)$, and let the conditions (A), (B) of Theorem 1 be satisfied. Further, let e be any fixed subset of the set e_n. Then the function $f^{(\nu)}(x) \in B_q^{(\rho)}(e \cap e_m; E^m)$ and the inequality

$$\|f^{(\nu)}(x)\|_{B_q^{(\rho)}(e\cap e_m;E^m)} \leq C\|f\|_{S_p^{(r)}B(E^n)} \tag{8}$$

holds.

Theorem 4. Let $f \in S_p^{(r)}B(E^n)$, where $r_i > 0$ $(i = 1, \dots, n)$, $p > 1$. Let natural numbers ν_i $(i = 1, \dots, n)$ and m $(0 < m \leq n)$, as well as numbers q $(p \leq q \leq \infty)$, ρ_s $(s = 1, \dots, m)$, be given, and suppose that the conditions (A), (B) of Theorem 1 are satisfied. Then, for any fixed $x_{m+1}, \dots, x_n$, the function $f^{(\nu)}(x) \in S_q^{(\rho)}B(E^m)$, and the inequality

$$\big\|f^{(\nu)}(x)\big\|_{S_q^{(\rho)}B(E^m)} \leq C\big\|f(x)\big\|_{S_p^{(r)}B(E^n)}. \tag{9}$$

holds.

We now introduce weighted spaces with fractional indices and formulate for them some embedding and extension theorems.

Let $f(x)$ be a smooth function given in the space $\bar{E}^{+n} = \{x = (x_1, \dots, x_n), x_n \geq 0\}$; $\Delta_i^2(t)f(x)$ is the finite difference of order 2 with respect to the variable x_i with step t. Let $1 \leq p < \infty$, $0 < r_i < 1$ $(i = 1, \dots, n)$. Introduce the following norms:

$$\|f(x)\|_{\mathcal{L}^{r_i}_{p,x_i,\alpha}(\bar{E}^{+n})} = \left(\int_0^\infty \frac{dt}{t^{1+pr_i}} \int_{\bar{E}^{+n}} x_n^\alpha \left|\Delta_i^2(t)f(x)\right|^p d\bar{E}^{+n}\right)^{1/p},$$

$$\|f(x)\|_{\mathcal{L}^{(r_1,\dots,r_n)}_{p,\alpha}(\bar{E}^{+n})} = \sum_{i=1}^n \|f(x)\|_{\mathcal{L}^{r_i}_{p,x_i,\alpha}(\bar{E}^{+n})},$$

$$\|f(x)\|_{B^{(r_1,\dots,r_n)}_{p,\alpha}(\bar{E}^{+n})} = \|f(x)\|_{L_p(\bar{E}^{+n})} + \|f(x)\|_{\mathcal{L}^{(r_1,\dots,r_n)}_{p,\alpha}(\bar{E}^{+n})}.$$

Definition 5. By the spaces $\mathcal{L}^{(r_1,\dots,r_n)}_{p,\alpha}(\bar{E}^{+n})$ and $B^{(r_1,\dots,r_n)}_{p,\alpha}(\bar{E}^{+n})$ we shall mean the closures of the set of smooth finite functions, respectively, in the norms $\|f\|_{\mathcal{L}^{(r_1,\dots,r_n)}_{p,\alpha}(\bar{E}^{+n})}$, $\|f\|_{B^{(r_1,\dots,r_n)}_{p,\alpha}(\bar{E}^{+n})}$.

Theorem 5. Let $p > 1$, $0 < r_i < 1$ $(i = 1, \dots, n)$, and let m be a natural number such that $0 < m < n$ and

$$\varepsilon = 1 - \frac{1}{p} \sum_{j=m+1}^n \frac{1}{r_j} - \frac{\alpha}{pr_n} > 0.$$

Then, if $f(x) \in B^{(r_1,\dots,r_n)}_{p,\alpha}(\bar{E}^{+n})$, then for any fixed $x_{m+1}, \dots, x_{n-1}$ and $x_n = 0$ the function $f(x) \in L_p(E^m)$, and the inequality

$$\|f\|_{L_p(E^m)} \leq C\|f\|_{B^{(r_1,\dots,r_n)}_{p,\alpha}(\bar{E}^{+n})}. \tag{10}$$

holds.

Theorem 6. Let $p > 1$, $0 < r_i < 1$ $(i = 1, \dots, n)$, and let m be a natural number such that $0 < m < n$, and suppose the conditions

$$\varepsilon = 1 - \frac{1}{p} \sum_{m+1}^n \frac{1}{r_i} - \frac{\alpha}{pr_n} > 0, \qquad 0 < \rho_s \leq \varepsilon r_s \quad (s = 1, \dots, m).$$

are satisfied. Then, if the function $f \in B^{(r_1,\dots,r_n)}_{p,\alpha}(\bar{E}^{+n})$, then for any fixed $x_{m+1}, \dots, x_{n-1}$ and $x_n = 0$ the function

$$f(x) \in \mathcal{L}_p^{(\rho_1,\dots,\rho_m)}(E^m),$$

and, moreover, the inequality

$$\|f\|_{\mathcal{L}_p^{(\rho_1,\dots,\rho_m)}(E^m)} \leq C\|f\|_{B_{p,\alpha}^{(r_1,\dots,r_n)}(\bar{E}^{+n})}, \tag{11}$$

holds; and if $\rho_s = \varepsilon r_s$ $(s = 1, \dots, m)$, then

$$\|f\|_{\mathcal{L}_p^{(\rho_1,\dots,\rho_m)}(E^m)} \leq C\|f\|_{\mathcal{L}_{p,\alpha}^{(r_1,\dots,r_n)}(\bar{E}^{+n})}. \tag{12}$$

As a consequence of Theorems 5 and 6 we obtain that, under the conditions of Theorem 6, if
$f \in B_{p,\alpha}^{(r_1,\dots,r_n)}(\bar{E}^n+)$, then for any fixed $x_{m+1}, \dots, x_{n-1}$ and $x_n = 0$ it also belongs to the space $B_p^{(\rho_1,\dots,\rho_m)}(E^m)$, and the inequality

$$\|f\|_{B_p^{(\rho_1,\dots,\rho_m)}(E^m)} \leq C\|f\|_{B_{p,\alpha}^{(r_1,\dots,r_n)}(\bar{E}^n+)} \tag{13}$$

holds.

Theorem 7. Let $p > 1$, $0 < r_i < 1$ $(i = 1, \dots, n)$, let m be a natural number such that $0 < m < n$, $\alpha > -1$, and

$$\varepsilon = 1 - \frac{1}{p} \sum_{j=m+1}^{n} \frac{1}{r_j} - \frac{\alpha}{pr_n} > 0.$$

Then, if on the hyperplane $x_{m+1} = \dots = x_n = 0$ a function
$\varphi(x_1, \dots, x_m) \in \mathcal{L}_p^{(\rho_1,\dots,\rho_m)}(E^m)$ is given, with $\rho_s = \varepsilon r_s$ $(s = 1, \dots, m)$, there exists a function
$f(x_1, \dots, x_n) \in \mathcal{L}_{p,\alpha}^{(r_1,\dots,r_n)}(\bar{E}^n+)$ such that

$$f\big|_{E^m} = \varphi, \qquad \|f\|_{\mathcal{L}_{p,\alpha}^{(r_1,\dots,r_n)}(\bar{E}^n+)} \leq C\|\varphi\|_{\mathcal{L}_p^{(\rho_1,\dots,\rho_m)}(E^m)}. \tag{14}$$

Theorem 8. Let $p > 1$, $0 < r_i < 1$ $(i = 1, \dots, n)$, let m be a natural number such that $0 < m < n$, $-1 < \alpha$, and

$$\varepsilon = 1 - \frac{1}{p} \sum_{i=m+1}^{n} \frac{1}{r_i} - \frac{\alpha}{pr_n} > 0.$$

Then, if on the hyperplane $x_{m+1} = \dots = x_n = 0$ a function
$\varphi(x_1, \dots, x_m) \in B_p^{(\rho_1,\dots,\rho_m)}(E^m)$ is given, with $\rho_s = \varepsilon r_s$ $(s = 1, \dots, m)$, there

exists a function
$f(x_1,\dots,x_n)\in B_{p,\alpha}^{(r_1,\dots,r_n)}(\bar{E}^n+\backslash\infty)$ such that

$$f\big|_{E^m}=\varphi,\qquad \|f\|_{B_{p,\alpha}^{(r_1,\dots,r_n)}(\bar{E}^n+\backslash\infty)}\le C\|\varphi\|_{B_p^{(\rho_1,\dots,\rho_m)}(E^m)}, \tag{15}$$

where
$\bar{E}^n+\backslash\infty=\{x\in\bar{E}^n+;\ x_i<\infty,\ i=m+1,\dots,n\}$.

Inequality (12) was obtained earlier by L. D. Kudryavtsev for $r_i=1$ ($i=1,\dots,n$), $m=n-1$, but under a more general assumption, i.e. there the p-summability of the function itself is not required. Inequality (14) was also proved earlier by L. D. Kudryavtsev for $r_i=1$ ($i=1,\dots,n$), $m=n-1$. A result close to inequality (9) was obtained simultaneously and independently of the author, by another method, by T. I. Amanov.

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