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Abstract

Full Text

MATHEMATICS

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ON SOME CHARACTERISTIC PROPERTIES OF UNCONDITIONAL BASES

(Presented by Academician A. N. Kolmogorov, November 29, 1963)

1. Let E be a Banach space over the field of complex numbers. It is easy to prove that for unconditionally convergent series the following holds.

Proposition 1. If the series $\sum_{k=1}^{\infty} x_k$ converges unconditionally, then the series

$$\sum_{k=1}^{\infty} |f(x_k)|$$

converges uniformly with respect to $f \in E^*$ and $\|f\| \leqslant 1$.

Let us agree to denote by the symbol $[x_k]$ the closure of the linear span of the system of elements $(x_k) \subset E$.

Proposition 2. In order that a basis (x_k) of the space E be unconditional, it is necessary and sufficient that, for every increasing sequence (n_k) of natural numbers, the subspace $[x_{n_k}]$ have a complement in the space E.

Proof. Necessity. Let (f_k) be the system of functionals biorthogonal to (x_k): $f_i(x_k) = \delta_{ik}$, $i, k = 1, 2, \ldots$. If (n_k) is an increasing sequence of natural numbers, then, by Orlicz' s theorem (^{1}), the series $\sum_{k=1}^{\infty} f_{n_k}(x) x_{n_k}$ converges for every $x \in E$, and, since (x_{n_k}) is a basis in $[x_{n_k}]$, this series converges to some $y \in [x_{n_k}]$:

$$\sum_{k=1}^{\infty} f_{n_k}(x) x_{n_k} = y = \sum_{k=1}^{\infty} \varphi_{n_k}(y) x_{n_k},$$

where the functionals φ_{n_k}, $k = 1, 2, \ldots$, are the restrictions of the corresponding functionals f_{n_k} to $[x_{n_k}] \subset E$. Thus every element $x \in E$ is represented, evidently uniquely, in the form of a sum:

$$x = y + z; \qquad y \in [x_{n_k}]; \qquad z = x - y \in [x_n]_{n \neq n_k}; \qquad E = [x_{n_k}] \oplus [x_n]_{n \neq n_k}. \tag{1}$$

Sufficiency. If for an arbitrary increasing sequence (n_k) of natural numbers (1) holds, then, obviously, $f_{n_k}(x) = f_{n_k}(y)$, and for every $x \in E$ the series

$\sum_{k=1}^{\infty} f_{n_k}(x)x_n$ converges, which, by the cited theorem of Orlicz, completes the proof.

Proposition 3. In order that a basis (x_k) be unconditional in the space E, it is necessary and sufficient that, for every x, the series

$$\sum_{k=1}^{\infty} |f_k(x)|\,|f(x_k)| \tag{2}$$

converge uniformly with respect to $f \in E^*$, $\|f\| \leqslant 1$.

The validity of this assertion follows from Proposition 1.

Corollary. If (x_k) is an unconditional basis in the space E, then for every $x \in E$ the series

$$\sum_{k=1}^{\infty} |f_k(x)|\, x_k$$

converges.

Remark. If (x_k) is a basis in E, then the convergence, for all x, of the series

$$\sum_{k=1}^{\infty} |f_k(x)|\, x_k$$

is not sufficient for the given basis to be unconditional. Indeed, consider the following example.

Example. Consider, in the space l of absolutely summable numerical sequences, the system of elements

$$f_k = (0, \dots, 0, 1, -1, 0, \dots), \qquad k = 1, 2, \dots$$

where the first nonzero entry is in the k-th position. This system is biorthogonal to the system (x_k) of elements of the space c_0:

$$x_k = (1, \dots, 1, 0, \dots), \qquad k = 1, 2, \dots,$$

where the last 1 is in the k-th position, which forms a basis in c_0. Hence the system (f_k) forms a basis in $[f_k] \subset l$. It is easy to see that $[f_k] \neq l$. Therefore the basis (x_k) in c_0 and the basis (f_k) in $[f_k]$ are not unconditional, since otherwise, taking into account the weak completeness of the space l, by Karlin' s theorem (^{2}), the system (f_k) would have to be a basis in the space l. Meanwhile, one can prove that if the series

$$\sum_{k=1}^{\infty} a_k x_k$$

converges, then the series

$$\sum_{k=1}^{\infty} |\alpha_k|\, x_k$$

also converges. The following, however, is true.

Theorem 1. For a basis (x_k) to be unconditional, it is necessary and sufficient that for every $x \in E$ the series

$$\sum_{k=1}^{\infty} |f_k(x)|\, x_k$$

converge and that the inequality

$$\left\| \sum_{k=1}^{\infty} f_k(x)\, x_k \right\| \le \left\| \sum_{k=1}^{\infty} |f_k(x)|\, x_k \right\|, \tag{3}$$

hold, where the constant M does not depend on the element $x \in E$.

Proof. Necessity. If the basis (x_k) is unconditional, then the corollary implies convergence of the series

$$\sum_{k=1}^{\infty} |f_k(x)|\, x_k.$$

If

$$\bar{f}_k(x) = \varepsilon_k |f_k(x)|; \qquad |\varepsilon_k| = 1; \qquad k = 1, 2, \dots,$$

then, applying the estimate of L. A. Gurevich (^{3}), we obtain

$$\left\| \sum_{k=1}^{\infty} f_k(x)\, x_k \right\| = \left\| \sum_{k=1}^{\infty} \varepsilon_k |f_k(x)|\, x_k \right\| \le M \left\| \sum_{k=1}^{\infty} |f_k(x)|\, x_k \right\|.$$

Sufficiency. Let (x_k) be a basis, and suppose that for every x the series

$$\sum_{k=1}^{\infty} |f_k(x)|\, x_k$$

converges and that estimate (3) holds. If $x \in E$ and $f \in E^*$, then by (ε_k) denote a sequence of complex numbers such that $|\varepsilon_k| = 1$,

$$\varepsilon_k f_k(x) f(x_k) = |f_k(x)|\, |f(x_k)|.$$

If $\varepsilon > 0$, then

$$\left\| \sum_{k=p}^{q} |f_k(x)|\, x_k \right\| < \frac{\varepsilon}{M\|f\|} \quad \text{for } q > p \ge N_\varepsilon.$$

Denote by y the sum $\sum_{k=p}^{q} \varepsilon_k f_k(x) x_k$. Then

$$\sum_{k=p}^{q} |f_k(x)|\, |f(x_k)| = \sum_{k=p}^{q} \varepsilon_k f_k(x) f(x_k) = f\left(\sum_{k=p}^{q} \varepsilon_k f_k(x) x_k \right) = f\left(\sum_{k=1}^{\infty} f_k(y) x_k \right) \le$$

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$$\leq \|f\| \left\| \sum_{k=1}^{\infty} f_k(y) x_k \right\| \leq M \|f\| \left\| \sum_{k=1}^{\infty} |f_k(y)| x_k \right\| = M \|f\| \left\| \sum_{k=p}^{q} |f_k(x)| x_k \right\| < \varepsilon$$

and, consequently, the basis (x_k) is unconditional.

Theorem 2. In order that the basis (x_k) be unconditional, it is necessary and sufficient that, for arbitrary n and x, the inequalities

$$m \left\| \sum_{k=1}^{n} |f_k(x)| x_k \right\| \leq \left\| \sum_{k=1}^{n} f_k(x) x_k \right\| \leq M \left\| \sum_{k=1}^{n} |f_k(x)| x_k \right\| \tag{4}$$

hold.

In the proof one uses Theorem 1 and an estimate of L. A. Gurevich (^{3}).

2. A system $(u_k) \subset E$ will be called **unconditionally ω-linearly independent** if, from the unconditional convergence of the series $\sum_{k=1}^{\infty} c_k u_k = \theta$ to the zero element $\theta \in E$, there follow the equalities $c_k = 0$; $k = 1, 2, \ldots$.

Definition. We shall say that a basis (x_k) of a space E is Φ-stable if every unconditionally ω-linearly independent system $(u_k) \subset E$ for which the series

$$\sum_{k=1}^{\infty} \|u_k - x_k\| f_k \tag{5}$$

converges is also a basis of the space E.

Theorem 3. In order that a basis $(x_k) \subset E$ be unconditional, it is necessary and sufficient that it be Φ-stable.

Proof. Necessity was proved by the author (^{4}) in the case where the space E is reflexive. If one takes into account Proposition 3, the proof also goes through for an arbitrary space.

Sufficiency. Suppose that the basis (x_k) is Φ-stable, but not unconditional. Then there exist $x_0 \in E$ and $f_0 \in E^*$ such that the series $\sum_{k=1}^{\infty} |f_k(x_0)|\,|f_0(x_k)|$ diverges. Let $|f_k(x_0)|\,|f_0(x_k)| = \varepsilon_k f_k(x_0) f_0(x_k)$. Denote by $\mathcal{L}$ the set of functionals that are linear combinations of the functionals (f_k) with nonnegative coefficients:

$$\mathcal{L} = \left\{ f; \quad f = \sum_{k=1}^{s} \beta_k f_k; \;\; \beta_k \geq 0 \right\}.$$

Let $\varphi \in \mathcal{L}$ and $\varphi(x_0) \neq 0$. Obviously, $\varphi(x_k) \geq 0$. Consider the system

$$y_k = \frac{1}{\varepsilon_k} x_k - \frac{x_0}{\varepsilon_k \varphi(x_0)} \varphi(x_k); \quad k = 1, 2, \ldots . \tag{6}$$

This system is unconditionally ω-linearly independent. Indeed, let

$$\sum_{k=1}^{\infty} c_k y_k = \theta$$

and let the series converge unconditionally. Then the series

$$\sum_{k=1}^{\infty} |f_0(c_k y_k)| = \sum_{k=1}^{\infty} |c_k| \, |f_0(y_k)|$$

converges, and for the chosen ε_k the series

$$\sum_{k=1}^{\infty} \varepsilon_k c_k f_0(y_k) = \sum_{k=1}^{\infty} c_k f_0(\varepsilon_k y_k)$$

converges absolutely.

From equality (6) it follows that the series

$$\sum_{k=1}^{\infty} c_k f_0(x_k) = \sum_{k=1}^{\infty} c_k f_0(\varepsilon_k y_k) + \frac{f_0(x_0)}{\varphi(x_0)} \sum_{k=1}^{s} c_k \varphi(x_k).$$

Let

$$\sum_{k=1}^{\infty} \frac{c_k}{\varepsilon_k} \varphi(x_k) = \sum_{k=1}^{s} \frac{c_k}{\varepsilon_k} \varphi(x_k) = \alpha.$$

Then

$$\sum_{k=1}^{\infty} \frac{c_k}{\varepsilon_k} x_k = \frac{x_0}{\varphi(x_0)} \sum_{k=1}^{\infty} \frac{c_k}{\varepsilon_k} \varphi(x_k) = \frac{\alpha x_0}{\varphi(x_0)}; \tag{7}$$

$$c_k = \varepsilon_k f_k \left(\frac{\alpha x_0}{\varphi(x_0)} \right) = \frac{\alpha}{\varphi(x_0)} \varepsilon_k f_k(x_0); \qquad k = 1, 2, \ldots ; \tag{8}$$

$$\sum_{k=1}^{\infty} c_k f_0(x_k) = \frac{\alpha}{\varphi(x_0)} \sum_{k=1}^{\infty} \varepsilon_k f_k(x_0) f_0(x_k) = \frac{\alpha}{\varphi(x_0)} \sum_{k=1}^{\infty} |f_k(x_0)| \, |f_0(x_k)|. \tag{9}$$

Equality (9) is possible only when $\alpha = 0$, and consequently, by (8), $c_k = 0$, $k = 1, 2, \dots$. Therefore the system (y_k), and together with it the system $(\varepsilon_k y_k)$, is unconditionally ω-linearly independent. And since the series

$$\sum_{k=1}^{\infty} \|\varepsilon_k y_k - x_k\| f_k = \sum_{k=1}^{\infty} \frac{\|x_0\|}{|\varphi(x_0)|} \varphi(x_k) f_k = \frac{\|x_0\|}{|\varphi(x_0)|} \sum_{k=1}^{s} \varphi(x_k) f_k$$

converges, by the assumption of the theorem the system $(\varepsilon_k y_k)$ forms a basis of the space E. The latter, however, is impossible, since

$$\varphi(\varepsilon_k y_k) = \varphi(x_k) - \frac{\varphi(x_0)}{\varphi(x_0)} \varphi(x_k) = 0; \qquad k = 1, 2, \dots,$$

and, consequently, the system $(\varepsilon_k y_k)$ is not even complete.

Remark. The convergence condition for the series (5) is weaker than the condition of M. G. Krein, M. A. Rutman, and D. P. Milman (5).

Theorem 4. *In order that a normalized basis (x_k) in Hilbert space be a Riesz basis* (6)*, it is necessary and sufficient that it be Φ-stable.*

The proof is based on Theorem 3 and on known theorems of N. K. Bari (6) and I. M. Gelfand (7).

Theorem 5. *If $\{x_k(t)\}$ is an unconditional basis in $L^p_{[a,b]}$ with biorthogonal system of functions $\{f_k(t)\} \subset L^q$ $(p^{-1}+q^{-1} = 1)$, then every ω-linearly independent system of functions $\{u_k(t)\} \subset L^p$ for which*

$$\int_a^b \left[\sum_{k=1}^{\infty} \|x_k - u_k\|^2_{(p)} \, |f_k(t)|^2 \right]^{q/2} dt < +\infty,$$

is also an unconditional basis in L^p, equivalent to the given one.

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Note: Figure translations are in progress. See original paper for figures.

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