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The number of approximations required in solving a system of linear equations

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Abstract

Full Text

MATHEMATICS

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IMPROVING THE CONVERGENCE OF METHODS OF SUCCESSIVE APPROXIMATIONS FOR LINEAR EQUATIONS

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The number of approximations required in solving a system of linear equations

$$\bar{x} = B\bar{x} + \bar{b} \tag{1}$$

by the method of successive approximations

$$\bar{x}_{n+1} = B\bar{x}_n + \bar{b} \tag{2}$$

may be large if the norm of the matrix B is close to 1, the initial approximation $\bar{x}_0$ is far from the solution, and the result must be obtained with high accuracy. Below a method is proposed for improving convergence, in which s approximations are equivalent to $n = 2^{s-1}$ ordinary approximations.

Solving equation (1) by successive approximations, we obtain:

$$\begin{aligned} \bar{x}_1 &= B\bar{x}_0 + \bar{b}, \\ \bar{x}_2 &= B^2\bar{x}_0 + (E + B)\bar{b}, \\ &\ldots \\ \bar{x}_n &= B^n\bar{x}_0 + (E + B + B^2 + \ldots + B^{n-1})\bar{b}. \end{aligned} \tag{3}$$

Transform the expression

$$\begin{aligned} U_n &= E + B + B^2 + \ldots + B^{n-1}, \\ U_n &= (E + B) + B^2(E + B) + B^4(E + B) + \ldots + B^{n-2}(E + B) = \\ &= (E + B^2 + B^4 + \ldots + B^{n-2})(E + B) = \\ &= ((E + B^2) + B^4(E + B^2) + \ldots + B^{n-4}(E + B^2))(E + B) = \\ &= (E + B^4 + B^8 + \ldots + B^{n-4})(E + B^2)(E + B). \end{aligned} \tag{4}$$

Continuing the process, we obtain

$$U_n = (E + B^{2^{s-2}})(E + B^{2^{s-3}}) \dots (E + B^2)(E + B). \tag{5}$$

Of course, in this case

$$n = 2^{s-1}. \tag{6}$$

Using relations (3), we obtain

$$\bar{x}_{2^{s-1}} = B^{2^{(s-1)}} \bar{x}_0 + (E + B^{2^{s-2}})(E + B^{2^{s-3}}) \dots (E + B^4)(E + B^2)(E + B)\bar{b} =$$

$$= B^{2^{s-1}} \bar{x}_0 + U_{2^{s-1}} \bar{b}. \tag{7}$$

The sequence of matrices U_{2^s-1} is easily computed, since

$$U_{2^s-1} = \left(E + B^{2^{s-2}}\right) U_{2^{s-2}}, \qquad U_1 = E; \tag{8}$$

$$B^{2^s-1} = \left(B^{2^{s-2}}\right)^2. \tag{9}$$

The same method gives a way of determining the inverse matrix

$$G = (E - B)^{-1},$$

$$G_{2^s-1} = B^{2^s-1} G_0 + U_{2^s-1}.$$

It should be noted that the method can also be applied to functional equations, provided it is possible to construct an effective process for computing the operators B^{2^s-1} and U_{2^s-1}.

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Note: Figure translations are in progress. See original paper for figures.