MATHEMATICS

1964

SovietRxiv

Abstract

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MATHEMATICS

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ON CONDITIONS FOR STRONG EQUIVALENCE OF GAUSSIAN MEASURES IN FUNCTION SPACE

(Presented by Academician A. N. Kolmogorov, 29 V 1964)

1. We shall consider Gaussian measures m_ξ in the space of functions $x(t) - x(0)$ $(0 \le t \le T)$, corresponding to real separable random processes $\xi(t)$ with stationary Gaussian increments, mean value 0, and spectral density $f_\xi(\lambda)$.

Let $x(t) - x(0)$ $(0 \le t \le T)$ be a sample function which is a realization of one of two processes: $\xi(t) - \xi(0)$ or $\eta(t) - \eta(0)$. We shall assume that $f_\eta(\lambda) \ge f_\xi(\lambda)$, so that one may put $f_\eta(\lambda) = f_\xi(\lambda) + f_\zeta(\lambda)$, where $f_\zeta(\lambda)$ is also a spectral density. Let x_n be the column vector of the increments of the function $x(t)$ on the intervals

$$\Delta t_j = \left[\frac{j-1}{n}T, \frac{jT}{n}\right]$$

$(j = 1, \dots, n)$. Denote by $p_\xi(x_n)$ the n-dimensional probability density of the vector x_n, corresponding to the process $\xi(t)$, and by $A_n(\xi)$ the covariance matrix of this vector. Then $A_n(\xi) = \|a_{jk}^{(n)}(\xi)\|$, where

$$a_{jk}^{(n)}(\xi) = \frac{4}{T}\int_{-\infty}^{\infty} e^{i\frac{k-j}{n}\lambda} \sin^2\frac{\lambda}{2n}\, f_\xi\left(\frac{\lambda}{T}\right) d\lambda \qquad (j, k = 1, \dots, n). \tag{1}$$

If $L(x_n) = p_\xi(x_n)/p_\eta(x_n)$ is the likelihood ratio, then in the Gaussian case

$$L(x_n) = \left\{\frac{|A_n(\eta)|}{|A_n(\xi)|}\right\}^{1/2} \exp\left\{-\frac{1}{2}x_n'\,[A_n^{-1}(\xi) - A_n^{-1}(\eta)]x_n\right\}. \tag{2}$$

We shall be interested in the behavior of the function $L(x_n)$ as $n \to \infty$. It is known that if the Gaussian measures m_ξ and m_η are orthogonal, then $L(x_n)$, as $n \to \infty$, with probability 1 tends either to zero or to $+\infty$. If, however, the measures m_ξ and m_η are equivalent, then $L(x_n)$ will, with probability 1, tend to a finite nonzero limit—the Radon-Nikodym derivative of the measure m_ξ with respect to the measure m_η (see, for example, ([1]), § 4.2). If, moreover, the limit $\lim_{n\to\infty} L(x_n)$ can be computed as the product of the limits of the two factors on the right-hand side of (2), i.e., if

$$\lim_{n\to\infty} L(x_n) = \lim_{n\to\infty} \left\{ \frac{|A_n(\eta)|}{|A_n(\xi)|} \right\}^{1/2} \lim_{n\to\infty} \exp\left\{ -\frac{1}{2} x_n' \left[A_n^{-1}(\xi) - A_n^{-1}(\eta) \right] x_n \right\}, \quad (3)$$

then, following Hajek ([2]) (see also ([3]), § 3), we shall call the measures m_ξ and m_η strongly equivalent; in this case the computation of the likelihood ratio for the processes is greatly simplified. Therefore it is of interest to find conditions for strong equivalence of measures. We shall denote strong equivalence of measures by the sign $\simeq$, ordinary equivalence by the sign $\sim$, and orthogonality by the sign $\perp$. If the measures m_ξ and m_η are not strongly equivalent, although perhaps equivalent in the ordinary sense, we shall write $m_\eta \not\simeq m_\xi$. The following two theorems give sufficient conditions under which $m_\eta \simeq m_\xi$ or $m_\eta \not\simeq m_\xi$.

Theorem 1. Let, for large $|\lambda|$,

$$f_\xi(\lambda) \geq c_1 |\lambda|^{-\alpha_1} \quad (c_1 > 0),$$

$$f_\zeta(\lambda) \leq c_2 |\lambda|^{-\alpha_2},$$

where $1 < \alpha_1 < \alpha_2 - 1$. Then $m_\eta \simeq m_\xi$.

Theorem 2. Let, for large $|\lambda|$,

$$f_\xi(\lambda) \leq c_1 |\lambda|^{-\alpha}, \qquad f_\zeta(\lambda) \geq c_2 |\lambda|^{-(\alpha+1)} \quad (c_2 > 0),$$

where $\alpha > 1$. Then $m_\eta \not\simeq m_\xi$.

From Theorem 1, in particular, there immediately follows Gaek' s result ([2]), which consists in the fact that for processes with rational spectral density equivalence of measures is always strong.

2. Using the equality $A_n(\eta) = A_n(\xi) + A_n(\zeta)$, let us represent the determinant of the matrix $A_n(\eta) A_n^{-1}(\xi)$ in the form

$$\begin{aligned} |A_n(\eta) A_n^{-1}(\xi)| &= |E_n + A_n(\zeta) A_n^{-1}(\xi)| \\ &= \exp \operatorname{Sp} \ln [E_n + A_n(\zeta) A_n^{-1}(\xi)], \end{aligned} \quad (4)$$

where E_n is the identity matrix of order n, and $\operatorname{Sp} A$ is the trace of the matrix A. Using the expansion in a series of the logarithmic function and the relation $\operatorname{Sp} |A_n(\zeta) A_n^{-1}(\xi)|^2 = O(1)$, which follows (see ([4]), Lemma 1) from the equivalence of the measures m_ξ and m_η, it is not difficult to show that the limit* $\lim_{n\to\infty} |A_n(\eta) A_n^{-1}(\xi)|$ will be finite if and only if

$$\operatorname{Sp} A_n(\zeta) A_n^{-1}(\xi) = O(1). \quad (5)$$

Thus, for the case $f_\eta(\lambda) \geq f_\xi(\lambda)$, the following criterion for strong equivalence of the Gaussian measures m_ξ and m_η holds.

Lemma 1. The relation (5) is a necessary and sufficient condition for the strong equivalence of the measures m_ξ and m_η.

Next, the following two lemmas hold.

Lemma 2. Let $f_{\eta_j}(\lambda) = f_{\xi_j}(\lambda) + f_{\zeta_j}(\lambda)$ $(j = 1, 2)$, with $f_{\xi_1}(\lambda) \leq f_{\xi_2}(\lambda)$ and $f_{\zeta_1}(\lambda) \geq f_{\zeta_2}(\lambda)$. Then from the relation $m_{\eta_1} \simeq m_{\xi_1}$ it follows that $m_{\eta_2} \simeq m_{\xi_2}$, and from $m_{\eta_2} \not\simeq m_{\xi_2}$ it follows that $m_{\eta_1} \not\simeq m_{\xi_1}$.

Lemma 3. Let $f_{\eta_1}(\lambda) = f_{\xi_1}(\lambda) + f_{\zeta_1}(\lambda)$, with the ratio $f_{\zeta_1}(\lambda)/f_{\xi_1}(\lambda)$ monotonically nonincreasing on the half-line $(0, \infty)$. Let, further, the function $\varphi(\lambda)$ be nonnegative, not identically equal to zero, and monotonically nondecreasing on the half-line $(0, \infty)$. If the function $f_{\eta_1}(\lambda)\varphi(\lambda)$ is the spectral density of some process $\eta_2(t) = \xi_2(t) + \zeta_2(t)$, where $f_{\xi_2}(\lambda) = f_{\xi_1}(\lambda)\varphi(\lambda)$ and $f_{\zeta_2}(\lambda) = f_{\zeta_1}(\lambda)\varphi(\lambda)$, then from the relation $m_{\eta_1} \simeq m_{\xi_1}$ it follows that $m_{\eta_2} \simeq m_{\xi_2}$, and from $m_{\eta_2} \not\simeq m_{\xi_2}$ it follows that $m_{\eta_1} \not\simeq m_{\xi_1}$.

We shall not give the proofs of Lemmas 2 and 3, since they are proved in exactly the same way as Lemmas 2 and 3 of ([4]).

3. **Proof of Theorem 1.** Introduce the auxiliary process $\chi(t)$, whose spectral density, for large $|\lambda|$, can be represented in the form

$$f_\chi(\lambda) = |\lambda|^{-\alpha} + o(|\lambda|^{-\alpha}),$$

where $\alpha = \frac{1}{2}(\alpha_1 + \alpha_2)$. The following obvious equality holds:

$$\mathrm{Sp}\, A_n(\zeta)A_n^{-1}(\xi) = \mathrm{Sp}\,[A_n^{-1/2}(\chi)A_n(\zeta)A_n^{-1/2}(\chi)][A_n^{1/2}(\chi)A_n^{-1}(\xi)A_n^{1/2}(\chi)]. \quad (6)$$

Hence, using the Cauchy–Bunyakovsky inequality, we find

$$\mathrm{Sp}\, A_n(\zeta)A_n^{-1}(\xi) \leq$$

$$\leq \sqrt{\mathrm{Sp}\ |A_n^{-1/2}(\chi)A_n(\zeta)A_n^{-1/2}(\chi)|^2}\, \sqrt{\mathrm{Sp}\ |A_n^{1/2}(\chi)A_n^{-1}(\xi)A_n^{1/2}(\chi)|^2}. \quad (7)$$

* The existence of this limit follows from Theorem 6.1 of ([2]).

Let $\theta_1(t)$ and $\theta_2(t)$ be processes with spectral densities, respectively, $f_{\theta_1}(\lambda) = f_\xi(\lambda) + f_\chi(\lambda)$ and $f_{\theta_2}(\lambda) = f_\chi(\lambda) + f_\zeta(\lambda)$. In accordance with Theorem 1 of paper [(4)], $m_{\theta_1} \sim m_\xi$ and $m_{\theta_2} \sim m_\chi$. Using now the criterion for equivalence of Gaussian measures (see [(4)], Lemma 1), we find

$$\mathrm{Sp}\,[A_n^{-1/2}(\chi)A_n(\xi)A_n^{-1/2}(\chi)]^2 = O(1), \tag{8}$$

$$\mathrm{Sp}\,[A_n^{1/2}(\chi)A_n^{-1}(\xi)A_n^{1/2}(\chi)]^2 = O(1). \tag{9}$$

From relations (7), (8), and (9) and Lemma 1, the validity of our theorem follows.

Theorem 2 is proved in exactly the same way as Theorem 2 of paper [(4)]. In the case $1 < \alpha \le 2$ we may, without loss of generality (by Lemma 2 and Theorem 1), assume that

$$f_\xi(\lambda) = \begin{cases} c_1\lambda^{-2}, & |\lambda| < 1, \\ c_1|\lambda|^{-\alpha}, & |\lambda| \ge 1; \end{cases}$$

$$f_\zeta(\lambda) = \begin{cases} 0, & |\lambda| < 1, \\ c_2|\lambda|^{-(\alpha+1)}, & |\lambda| \ge 1. \end{cases}$$

Using further Szegő' s theorem ([(5)], Theorem XIX) on the limiting distribution of the eigenvalues of a pair of Toeplitz forms, we directly obtain

$$\lim_{n\to\infty} \mathrm{Sp}\,A_n(\xi)A_n^{-1}(\zeta) = \infty,$$

whence, in accordance with Lemma 1, it follows that $m_\eta \ne m_\xi$. With the aid of Lemma 3 this result is immediately extended to the case $\alpha > 2$.

4. We give one more theorem, which does not follow from the preceding results.

Theorem 3. Let $f_\eta(\lambda) = f_\xi(\lambda) + f_\zeta(\lambda)$ and $f_{\eta_1}(\lambda) = f_\eta(\lambda) + f_\xi(\lambda)$, and suppose $m_\eta \ne m_\xi$. Then $m_{\eta_1} \ne m_\eta$.

Proof. If $m_\eta \perp m_\xi$, then, as shown in paper [(6)], $m_{\eta_1} \perp m_\eta$. Suppose now that $m_\eta \sim m_\xi$, but $m_\eta \ne m_\xi$. We shall show that

$$\lim_{n\to\infty} \mathrm{Sp}\,A_n(\xi)[A_n(\xi) + A_n(\zeta)]^{-1} = \infty. \tag{10}$$

Indeed,

$$\mathrm{Sp}\,A_n(\xi)[A_n(\xi) + A_n(\zeta)]^{-1} =$$

$$= \mathrm{Sp}\,A_n(\xi)A_n^{-1}(\xi)\{[A_n(\xi) + A_n(\zeta)]A_n^{-1}(\xi)\}^{-1} =$$

$$= \operatorname{Sp} A_n(\xi) A_n^{-1}(\xi)[E_n + A_n(\zeta) A_n^{-1}(\xi)]^{-1}.$$

Since $m_\eta \sim m_\xi$, all eigenvalues of the matrices $A_n(\zeta)A_n^{-1}(\xi)$ $(n = 1, 2, ...)$ are uniformly (in n) bounded. Therefore, from the unboundedness of the sequence $\operatorname{Sp} A_n(\xi)A_n^{-1}(\xi)$ there follows the unboundedness of the sequence $\operatorname{Sp} A_n(\xi)A_n^{-1}(\xi)[E_n + A_n(\zeta)A_n^{-1}(\xi)]^{-1}$. The validity of the theorem now follows from relation (10) and Lemma 1.

In conclusion the author expresses his gratitude to A. M. Yaglom for posing the problem.

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Received
22 V 1964

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Note: Figure translations are in progress. See original paper for figures.