OPERATORS OF POTENTIAL TYPE FOR THE CASE OF SETS WITH AN ABSTRACT MEASURE

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Abstract

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OPERATORS OF POTENTIAL TYPE FOR THE CASE OF SETS WITH AN ABSTRACT MEASURE

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1. Let R be a complete metric space with metric r_{ts}, and let Ω, Ω^* be bounded sets in R, on each of which some abstract absolutely continuous measure is given. The operator

$$Ax(t) = \int_{\Omega} r_{ts}^{-\lambda} x(s)\, ds \qquad (t \in \Omega^*), \tag{1}$$

following (1), is usually called an operator of potential type. For the study of such operators the following theorem of L. V. Kantorovich (^{2}) is convenient.

Theorem 1. Let the function $K(t, s)$ satisfy the conditions:

$$\int_{\Omega} |K(t, s)|^{\alpha}\, ds \leqslant C_1 \tag{2}$$

for almost all $t \in \Omega^*$;

$$\int_{\Omega^*} |K(t, s)|^{\beta}\, dt \leqslant C_2 \tag{3}$$

for almost all $s \in \Omega$, where $\alpha, \beta > 0$ and at least one of the numbers α, β is not less than one.

Then the operator

$$Kx(t) = \int_{\Omega} K(t, s) x(s)\, ds \tag{4}$$

is a continuous operator acting from $\mathcal{L}_p$ to $\mathcal{L}_q$, for any p and q satisfying the relations

$$\max(p, \beta) \leqslant q \leqslant \frac{\beta p}{\alpha + (1-\alpha)p}. \tag{5}$$

Verification of conditions (2) and (3) for the kernel $K(t,s) = r_{ts}^{-\lambda}$ requires knowledge of the relation between the metric of the space R and the measures given on the sets Ω and Ω^*. In the case when R is a finite-dimensional Euclidean space and the measures given on Ω and Ω^* are Lebesgue measures, this relation is expressed by means of the dimension of the sets, namely: if the dimension of the set Ω is equal to n, then

$$\int_\Omega r_{ts}^{-\mu}\, ds \leqslant C(\mu) < \infty \tag{6}$$

for any $\mu < n$. For other measures, even in Euclidean space, such a relation becomes more complicated (for example, for sets on curvilinear surfaces); establishing the relation in the abstract case is still more difficult. The reason lies in the fact that the abstract notion of the dimension of a set is in no way connected with the measure given on this set.

Below a notion of dimension is proposed that directly relates the metric and the measure on the set.

2. Denote by $\Omega_{s,\rho}$ the intersection of the set Ω and the ball of radius ρ with center at the point s. We shall call **admissible** for the set Ω any number $N > 0$ for which there exists a constant $C(N)$ such that

$$\operatorname{mes} \Omega_{s,\rho} \leq C(N)\rho^N \tag{7}$$

for almost all $s \in \Omega$. The **dimension** of the set Ω (with respect to the measure) is defined to be the exact upper bound of the numbers admissible for Ω; denote it by $N(\Omega)$. At the same time, $N(\Omega)$ itself may turn out to be either an admissible or a non-admissible number for Ω.

Let us note several simple properties of the concept introduced. It is easy to see that in the case of n-dimensional Lebesgue measure, from $\operatorname{mes}\Omega \neq 0$ it follows that $N(\Omega) = n$, and $N(\Omega)$ itself is an admissible number. It is also easily verified that $N(\Omega) \geq N(\Delta)$, if $\Delta \subset \Omega$; on the other hand, for any number N not admissible for Ω there will be found a set $\Delta \subset \Omega$, $\operatorname{mes}\Delta \neq 0$, and a constant C such that

$$\operatorname{mes} \Omega_{s,\rho} \geq C\rho^N \tag{8}$$

for all $s \in \Delta$.

The connection between the metric of the space R and the measure given on the set Ω is now expressed by the following assertion.

Lemma 1. *If N is a number admissible for Ω and $0 < \mu < N$, then for almost all $t \in \Omega$*

$$\int_{\Omega} r_{ts}^{-\mu}\, ds \leq C d^{N-\mu}, \tag{9}$$

where d is the diameter of the set Ω.

Let us note that the constant C in (9) depends only on $C(N)$ from inequality (7) and on the number μ.

3. Let us return to the consideration of operator (1). If $\inf r_{ts} > 0$ over all $t \in \Omega^*$, $s \in \Omega$, then the operator A has a bounded kernel and, consequently, is completely continuous in any pair of spaces $\mathcal{L}_p, \mathcal{L}_q$. On the other hand, different parts of some set may have different dimensions, possibly exceeding the dimension of the set itself. Thus, the properties of the operator are connected precisely with the properties of the intersection

$$\overline{\Omega} = \Omega \cap \Omega^*.$$

Introduce the notation

$$\Omega_\rho = \bigcup_{s \in \overline{\Omega}} \Omega_{s,\rho}, \qquad \Omega_\rho^* = \bigcup_{t \in \overline{\Omega}} \Omega_{t,\rho}^*, \qquad n = \sup_\rho N(\Omega_\rho), \qquad m = \sup_\rho N(\Omega_\rho^*).$$

The numbers n, m will determine the properties of the operator A.

Theorem 2. *The operator A maps $\mathcal{L}_p$ into $\mathcal{L}_q$ for any p and q satisfying one of the following relations:*

a) $\lambda < n, \quad p > \dfrac{n}{n-\lambda}, \quad q$ arbitrary;

b) $\lambda < n, \quad \dfrac{n-m}{n-\lambda} < p \leq \dfrac{n}{n-\lambda}, \quad q < \dfrac{mp}{n-(n-\lambda)p}$;

c) $n \leq \lambda < m, \quad p < \dfrac{m-n}{\lambda - n}, \quad q < \dfrac{mp}{n-(n-\lambda)p}$.

The theorem is proved by combining Lemma 1 and Theorem 1. From the general properties of integral operators (3) it follows that the operator A is even completely continuous in the spaces under consideration.

4. In the study of an operator of potential type there arises the question of estimating iterated kernels; in particular, it is necessary to estimate the function

$$K(t,s) = \int_{\Omega} r_{t\tau}^{-\lambda} r_{\tau s}^{-\mu}\, d\tau \qquad (\lambda, \mu < N(\overline{\Omega})). \tag{10}$$

Theorem 3. *Let* N_1 *be a number admissible for* $\overline{\Omega}$*, and let* $\lambda + \mu > N_1$*. Then* $K(t, s) \leq C_1 r_{ts}^{N_1 - \lambda - \mu}$ *for all* $t \in \Omega^*, \ s \in \Omega$*.*

Theorem 4. Let N_2 be a number inadmissible for $\overline{\Omega}$, and let $\lambda + \mu > N_2$.

Then there exists a set $\Delta \subset \overline{\Omega}$, $\operatorname{mes} \Delta \neq 0$, such that $K(t, s) \geq C_2 r_{ts}^{N_2 - \lambda - \mu}$ for all $t, s \in \Delta$.

The numbers N_1 and N_2 may be chosen arbitrarily close to one another. Therefore the results presented almost coincide with the known results for Lebesgue measure. The coincidence will be complete if the set $\overline{\Omega}$ is homogeneous with respect to dimension, i.e., if there exists a number N such that for almost all $s \in \overline{\Omega}$

$$C_3 \rho^N \leqslant \operatorname{mes} \overline{\Omega}_{s,\rho} \leqslant C_4 \rho^N. \tag{11}$$

5. The results of Theorem 2 are valid for an arbitrary integral operator A with kernel satisfying, for almost all t, s, the inequality $|K(t, s) r_{ts}^{\lambda}| \leqslant C < \infty$. Therefore Theorem 2 may be regarded as relating the location and magnitude of the singularities of the kernel to the spaces in which the corresponding operator acts. One may pose the problem of elucidating inverse relations among these three factors. Apparently, it is most difficult to draw nontrivial conclusions concerning the location of the singularities; however, knowing the location of the singularities of the kernel and the spaces in which the corresponding operator acts, one can often indicate the magnitude of the singularities of the kernel.

A set $D_0 \subset \Omega \times \Omega^*$ will be called a **rectangle with sides** a, b if $D_0 = \Delta \times \Delta^*$, $\Delta \subset \Omega$, $\Delta^* \subset \Omega^*$, $\operatorname{mes} \Delta = a$, $\operatorname{mes} \Delta^* = b$. For an arbitrary set D, by $\varphi_D(a, b)$ we denote the minimal number of nonoverlapping closed rectangles with sides a, b whose union contains D; this union itself will be denoted by D^{ab}.

Consider a function $K(t, s) \geq 0$; by E_σ denote the set of points $\{t, s\} \in \Omega \times \Omega^*$ at which $K(t, s) > \sigma$.

Theorem 5. Let $k, i \geq 0$, and suppose that for every σ there exists an ε such that simultaneously

$$\operatorname{mes} E_\sigma^{\varepsilon^k \varepsilon^i} \leqslant 2 \operatorname{mes} E_\sigma, \qquad \varphi_{E_\sigma}(\varepsilon^k, \varepsilon^i) \leqslant C \varepsilon^{-\lambda}. \tag{12}$$

Suppose moreover that the operator A with kernel $K(t, s)$ acts from $\mathcal{L}_p$ to $\mathcal{L}_q$ $(q \geq p)$.

Then the kernel $K(t, s)$ is summable with respect to the aggregate of variables with any degree

$$\alpha < \frac{k + i - \lambda}{k/p' + i/q}. \tag{13}$$

In particular, the results obtained in [4] follow from this.

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Note: Figure translations are in progress. See original paper for figures.

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