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Abstract

Full Text

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ON SMOOTH MAPPINGS OF THE SPACE R_n INTO R_1

(Presented by Academician A. N. Kolmogorov, November 29, 1963)

Let R_n be n -dimensional real Euclidean space, $\mathbf{x}_1, \dots, \mathbf{x}_n$ its fixed orthonormal basis; x^1, \dots, x^n the coefficients in the expansion of the vector $\overline{(0, x)}$ with respect to this basis; $\rho(E_1, E_2)$ the distance between the sets E_1 and E_2 ; dE the diameter of the set E ; $S(a, r) = \{x : \rho(x, a) \leq r\}$; (E, x) the connected component of the set E containing the point x ; $|E|$ the length of the set E ; $\dim E$ the dimension of the set E ; $d_k(E)$ ($k = 1, \dots, n-1$) the maximum of the measures of the projections of the set E onto all possible k -dimensional coordinate planes; $d_{nE} = \text{mes } E$; $E(f, t) = \{x : f(x) = t\}$; $k(f, t)$ the number of connected components of the set $E(f, t)$. Some quantities will be denoted by the letters a_0, a_1, \dots ; in parentheses we shall indicate the numbers on which they depend. Let s be a nonnegative integer, $0 \leq \alpha \leq 1$, $l = s + \alpha > 0$, and let $f(x)$ be a function mapping the space R_n into R_1 , equal to zero outside the ball $S(0, R)$, and having all derivatives up to order s inclusive. Suppose that the derivatives of the last order $f_i(x)$ satisfy the condition

$$|f_i(x) - f_i(y)| \leq M[\rho(x, y)]^\alpha. \quad (1)$$

Put $N = \{x : |\text{grad } f(x)| = 0\}$; $N_1 = \{x : (E(f, f(x)), x) \cap N \neq \emptyset\}$; $G = CN_1$. The components K_i of the open set G will be called **rings**.

Lemma 1. The boundary of a ring ∂K_i consists of two connected components F_i, H_i , one of which, say H_i , separates the other from infinity. The function $f(x)$ maps the ring K_i onto some interval (a_i, b_i) in such a way that the set $K_i \cap E(f, t)$ is connected for $t \in (a_i, b_i)$.

For the proof see (1), p. 136.

The set F_i will be called the **inner boundary** of the ring K_i , and H_i the **outer boundary**. Let $h_i = |b_i - a_i|$ be the length of the image of the ring K_i under the mapping by the function $f(x)$, $m_i = h_i^{1/l}$, $n_i = m_i^n$.

Theorem 1. Under the conditions on the function $f(x)$ stated above, the following estimate holds:

$$\sum_{K_i} n_i \leq a_1(n, l) M^{n/l} R^n. \quad (2)$$

Theorem 1 was proved by A. G. Vitushkin in two cases: 1) $n = 1$ and 2) $l = n$.

Theorem 2. Under the conditions of Theorem 1, if $l \geq n$, then

$$\int_{-\infty}^{\infty} [k(f, t)]^{l/n} dt \leq [a_1(n, l)]^{l/n} MR^l. \quad (3)$$

There is an example of a function $f_l(x)$ satisfying the conditions of Theorem 1 and such that, for every $\varepsilon > 0$, the series $\sum_{K_i} n_i^{1-\varepsilon}$ and the integral $\int_{-\infty}^{\infty} [k(f, t)]^{l/n+\varepsilon} dt$ diverge ...

...This example shows that Theorems 1 and 2 cannot be improved. We now prove Theorem 1.

Lemma 2 (Oleinik). *If $f(x)$ is a polynomial of degree s , then for any t*

$$k(f, t) \leq a_0(n, s). \quad (4)$$

Lemma 2'. *Let $f(x)$ be a polynomial of degree s , constant on the boundary of each of N domains G_i such that $G_i \cap G_j = 0$ for $i \neq j$. Then either $f(x) \equiv \text{const}$ or $N \leq a'_0(n, s)$.*

Lemma 3. *Let $f(x)$ be a polynomial of degree s , $E = (E(f, 0) \cap S(a, r), x)$. Let $y \in E$, $\varepsilon > 0$. There exist no more than $a_2(n, s)$ points x_i and polynomials $f_i(x)$, the degree of each of which does not exceed $a_3(n, s)$, such that: a) $f_i(x_i) = 0$; $x_i \in S(a, r)$; b) $\dim E_i = 1$, where $E_i = (E(f_i, 0) \cap S(a, r), x_i)$; c) $x \in \bigcup_i E_i$; $y \in \bigcup_i E_i$; d) the set $\bigcup_i E_i$ is connected; e) if $z \in E_i$, then $\rho(z, E) \leq \varepsilon$.*

Lemma 4. *Let $f(x)$ be a polynomial of degree s , $E_t = \{x : |f(x)| \leq t\}$. If $x \in E_t$, $y \in E_t$, $(E_t \cap S(a, r), x) = (E_t \cap S(a, r), y)$, then the points x and y can be joined by a curve γ such that $\gamma \in (E_t \cap S(a, r), x)$; $|\gamma| \leq a_4(n, s)r$.*

Proof of Theorem 1. If $l \leq 1$, then the theorem is obvious, since in each ring K_i there is a point A_i such that

$$S\left(A_i, \frac{1}{2} \left(\frac{h_i}{2M}\right)^{1/l}\right) \subset K_i.$$

Let $l > 1$, and let x_i be a point of the inner boundary of the ring K_i at which $|\text{grad } f(x)| = 0$. Put

$$\tilde{G}_{\varepsilon, t}^i = S(x_i, 2^t m_i) \cap \left\{x : |\text{grad } f(x)| \leq \frac{\varepsilon t}{n} m_i^{l-1}\right\}; \quad G_{\varepsilon, t}^i = (\tilde{G}_{\varepsilon, t}^i, x_i). \quad (5)$$

We divide the collection of rings K_i into $n + 1$ classes σ_k in the following way: $K_i \in \sigma_1$ if $dG_{\varepsilon, 1}^i \leq 0.1 n^{-2} m_i$. For $2 \leq k \leq n$, $K_i \in \sigma_k$ if $d_{k-1} G_{\varepsilon, k-1}^i > (0.1 n^{-2} m_i)^{k-1}$, but $d_{kG_{\varepsilon, k}}^i \leq (0.1 n^{-2} m_i)^k$, and $K_i \in \sigma_{n+1}$ if

$d_{nG_{\varepsilon,n}}^i > (0.1 n^{-2} m_i)^n$. Let the ring $K_i \in \sigma_1$. Put $S_i = S(y_i, m_i)$, where y_i is some point satisfying the condition $\rho(x_i, y_i) = 2m_i$.

Let the ring $K_i \in \sigma_k$, $2 \leq k \leq n$. Take numbers j_1, \dots, j_{k-1} such that the measure of the projection of the set $G_{\varepsilon, k-1}^i$ onto the $(k-1)$ -dimensional plane Π_i , spanned by the vectors $x_{j_1}, \dots, x_{j_{k-1}}$, exceeds $(0.1 n^{-2} m_i)^{k-1}$. One can find a set E^i of points of the plane Π_i such that $\text{mes } E^i \geq \frac{1}{2} (0.1 n^{-2} m_i)^{k-1}$, and the $(n-k+1)$ -dimensional plane $\Pi_{i,x}$, orthogonal to the plane Π_i and such that $\Pi_i \cap \Pi_{i,x} = x \in E^i$, contains a point $y_i(x) \in G_{\varepsilon, k-1}^i$ and $d(G_{\varepsilon, k-1}^i \cap \Pi_{i,x}, y_i(x)) \leq \frac{1}{5} m_i$. In this case the points $z_i(x)$ can be chosen so that: 1) $z_i(x) \in \Pi_{i,x}$, 2) $\rho(y_i(x), z_i(x)) = 2m_i$, and 3) the set

$$S_i = \bigcup_{x \in E^i} S(z_i(x), m_i) \cap \Pi_{i,x}$$

is measurable. If $K_i \in \sigma_{n+1}$, let

$$S_i = \{x : |x^k - x_i^k| \leq m_i, k = 1, \dots, n\}.$$

It is clear that, in order to prove the theorem, it is enough to estimate $\sum_i \text{mes } S_i$.

In doing so one may assume that the numbers ε and M are sufficiently small, say

$$\varepsilon < 2^{-4l-n-1} [a_4(n, 2s)]^{-1}, \quad M = 12^{-2l-1} n^{-2} \varepsilon.$$

Lemma 5. *Let the rings K_i and K_j be such that $h_j \leq h_i \leq 2^{2l} h_j$. Then $G_{\varepsilon,n}^i \cap G_{\varepsilon,n}^j = 0$.*

Indeed, otherwise, using Lemma 4, we would join the points x_i and x_j by a curve γ such that: 1) $|\gamma| \leq a_4(n, 2s)(2^{2l+n} + 2^n)m_i$, and 2) for $x \in \gamma$,

$$|\text{grad } f(x)| \leq \varepsilon 2^{2l} m_i^{l-1}.$$

This contradicts the inequality

$$\max_{x \in \gamma} f(x) - \min_{x \in \gamma} f(x) \geq h_j.$$

Lemma 6. *Let the rings $K_{i_r} \in \sigma_k$ for one and the same k , $1 \leq k \leq n$. Suppose moreover that, if $k \geq 2$, the planes Π_i coincide.*

There is no point common to more than $a_5(n, l)$ of the sets S_{i_r} .

Proof. Consider the case $k = 1$. Suppose that, contrary to the assertion of Lemma 6, the point $t \in S_{i_r}$ for $r = 1, \dots, a_5(n, l) + 1$.

From Lemma 5 and the construction of the sets S_{i_r} it follows that, for $r_1 \neq r_2$, $G_{\varepsilon,1}^{i_{r_1}} \cap G_{\varepsilon,1}^{i_{r_2}} = 0$. Therefore one can find either $a_0(n, 2s) + 1$ sets $G_{\varepsilon,1}^{i_r}$ contained one in another, or $a'_0(n, 2s) + 1$ sets $G_{\varepsilon,1}^{i_r}$ lying outside one another. Let $P(k, t, x)$ be a segment of the Taylor series of the function $\frac{\partial}{\partial x^k} f(x)$ with center at the point t , and

$$P(t, x) = \sum_{k=1}^n P^2(k, t, x).$$

In the first of the resulting cases we have $h_j \geq 2^{-2l} h_i$, where h_j is the minimum and h_i the maximum of the numbers h_{i_r} in the system under consideration. Using the estimate for the number M , we obtain:

$$k \left(P(t, x), \left[\frac{1}{2} \frac{\varepsilon}{n} 2^{-2l} m_i^{l-1} \right]^2 \right) \geq a_0(n, 2s) + 1,$$

which contradicts Lemma 2. In the second case the polynomial $P(t, x)$ remains constant on the boundary of $a'_0(n, 2s) + 1$ domains H_{i_r} such that $H_{i_r} \subset G_{\varepsilon,1}^{i_r}$. This contradicts Lemma 2'. Lemma 6 is proved in the case $k = 1$. The case $2 \leq k \leq n$ is treated analogously.

Lemma 7. If $K_i \in \sigma_{n+1}$, then $S_i \in G_{\varepsilon, a_6(n,l)}^i$. If $\rho(x, x_i) \geq m_i$, then

$$|\text{grad } f(x)| \leq M a_7(n, l) [\rho(x, x_i)]^{l-1}.$$

Let the number ε also satisfy the two inequalities

$$12^{-2l} n^{-2} a_7(n, l) \varepsilon < 2^{-2l}, \tag{6}$$

$$2^{l+1} \varepsilon a_6(n, l) < 1. \tag{7}$$

We divide the totality σ_{n+1} of the annuloids K_i into two parts $\sigma'_{n+1} \cup \sigma''_{n+1}$ in the following way: $K_i \in \sigma''_{n+1}$ if $\text{mes } K_i \geq 0.1 n_i$, and $\sigma'_{n+1} = \sigma_{n+1} \setminus \sigma''_{n+1}$. The estimate

$$\sum_{K_i \in \sigma''_{n+1}} \text{mes } S_i$$

is obvious. Let us estimate

$$\sum_{K_i \in \sigma'_{n+1}} \text{mes } S_i.$$

Choose some direction, for example that of the vector x_1 , and let Π be the plane orthogonal to it, while R_x is the line orthogonal to Π such that $\Pi \cap R_x = x$. Define the function $\chi_i(y)$ as follows: if $K_i \in \sigma'_{n+1}$, $y \in R_x \cap S_i$, $\rho(x_i, R_x) \leq m_i$, and from the relations $a_i \in F_i$, $b_i \in H_i$, $(R_x \cap \overline{K}_i, a_i) = (R_x \cap \overline{K}_i, b_i)$ there follows the inequality

$$|(R_x \cap \overline{K}_i, a_i)| \leq \frac{1}{2} m_i,$$

then $\chi_i(y) = 1$. In the opposite case $\chi_i(y) = 0$. Let

$$\chi(y) = \sum_i \chi_i(y).$$

In the obvious way the estimate

$$\sum_{K_i \in \sigma'_{n+1}} \text{mes } S_i$$

is reduced to estimating

$$\int_{R_n} \chi(y),$$

and the latter to estimating

$$\int_{R_x} \chi(y) dy.$$

We divide the totality σ'_{n+1} of the annuloids K_i into the following parts:

$$\sigma'_{n+1} = \bigcup_{k=1}^{\infty} \sigma'_{n+1,k},$$

and $K_i \in \sigma'_{n+1,k}$ if

$$4^{1-k}H \geq m_i > 4^{-k}H,$$

where

$$H = \max_i m_i.$$

Put

$$\Delta_i = \{y : (\chi_i(y) = 1) \cap R_x\}.$$

From Lemma 7 and relation (7) it follows that

Lemma 8. If $K_i \in \sigma'_{n+1,k}$, $K_j \in \sigma'_{n+1,k}$, then for $i \neq j$

$$\Delta_i \cap \Delta_j = 0.$$

Put

$$\chi^k(y) = \sum_{K_i \in \sigma'_{n+1,k}} \chi_i(y).$$

We have

$$\chi(y) = \sum_{k=1}^{\infty} \chi^k(y) = \sum'_k \chi^k(y) + \sum''_k \chi^k(y),$$

where \sum'_k is taken over those k for which there exist at least two numbers i_1, i_2 such that, for $y \in R_x$, $\chi_{i_1} \neq 0$, $\chi_{i_2} \neq 0$, and $K_{i_1} \in \sigma'_{n+1,k}$, $K_{i_2} \in \sigma'_{n+1,k}$. Put

$$\chi_1^*(y) = \sum'_k \chi^k(y), \quad \chi_2^*(y) = \sum''_k \chi^k(y).$$

The estimate

$$\int_{R_x} \chi_2^*(y) dy$$

is obvious. Let, for $y \in R_x$,

$$\chi_1^*(y) = \sum_{k \in E} \chi^k(y) = \sum_{k \in E} \sum_{i \in E_k} \chi_i(y).$$

Put

$$k \in E; \quad D_k = \{y : \chi^k(y) = 1\} \cap R_x,$$

and let Δ_i^k be the connected components of the set D_k . From the definition of the collection σ'_{n+1} it follows that between two adjacent intervals Δ_i^k, Δ_j^k there exists a point z at which

$$|\text{grad } f(z)| \geq 4^{-(l-1)k} H^{(l-1)k}.$$

Put

$$\Delta_{k,j} = R_x \cap S(z, 2^{-k}H).$$

From Lemma 7 and estimates (6) and (7) the following lemmas follow:

Lemma 9. If $k_1 \ll k_2$, then

$$\Delta_{i,j}^{k_1} \cap \Delta_{i,j}^{k_2} = 0.$$

Lemma 10. The intervals $\Delta_{i,j}^k$, for one and the same k , are pairwise disjoint.

Now the proof of Theorem 1 is completed as follows. Each of the intervals $\Delta_{i,j}^{k_1}$ can intersect no more than one of the intervals $\Delta_{i,j}^{k_2}$ for $k_2 > k_1$. Hence we obtain the estimate

$$\int_{R_x} \chi_1^*(y) dy \leq 2 \sum_{i,j,k} |\Delta_{i,j}^k| \leq 8R.$$

Integrating this inequality over the plane Π , we obtain the required result. Theorem 1 is proved.

Theorem 2 is easily obtained from Theorem 1, Lemma 1, and the theorem of E. M. Landis, which asserts that if $l \geq n$, then

$$\text{mes}\{t : t = f(x), x \in N\} = 0.$$

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REFERENCES

1. A. G. Vitushkin, *On multidimensional variations*, Moscow, 1955.

Note: Figure translations are in progress. See original paper for figures.

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