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1964

SovietRxiv

Abstract

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A THEOREM OF THE TYPE OF THE PHRAGMÉN-LINDELÖF THEOREMS FOR HARMONIC FUNCTIONS IN SPACE

(Presented by Academician P. S. Novikov, 28 XII 1963)

In the paper ([1]), M. A. Evgrafov and I. A. Chegis established the following theorem:

Let $u(\rho, \varphi, t)$ be a harmonic function in the circular cylinder $0 \leq \rho \leq a,\ 0 \leq \varphi < 2\pi,\ -\infty < t < +\infty$. If the conditions

$$u(a, \varphi, t) = 0, \qquad \left|\frac{\partial u}{\partial \rho}(a, \varphi, t)\right| < c,$$

$$\max_{(\rho,\varphi)} |u(\rho, \varphi, t)| < c_1 \exp\exp\frac{\pi|t|}{2(a+\varepsilon)}, \qquad \varepsilon > 0,$$

are fulfilled, then $u(\rho, \varphi, t) \equiv 0$.

An analogous theorem was established by I. A. Chegis in the paper ([2]) for the case of a cylinder with a rectangular base. In the proofs, the apparatus of the theory of Dirichlet series or else the Fourier transform is used. In the present note, for the proof of theorems of a similar type, a method is proposed that is based only on properties of subharmonic functions; this method requires, on the one hand, somewhat greater restrictions, but, on the other hand, makes it possible to obtain certain generalizations as well. For the case of the plane, the method was set forth in ([3]).

Theorem. Let D be a domain in the space (x, y, t), contained in the cylinder $-a \leq x \leq a,\ -b \leq y \leq b,\ -\infty < t < +\infty$, and suppose that the boundary Γ of the domain D consists of smooth surfaces. Let $u(x, y, t)$ be a nonnegative harmonic function in the enlarged domain D_1, containing every ball with center at a point $P \in D$ of radius δ, where $\delta > 0$ is a fixed number. Further, suppose that for u the conditions

$$u(x, y, t)\big|_{\Gamma} \leq M, \tag{1}$$

$$\max_{(x,y)} u(x, y, t) < c\exp\exp\frac{\pi(1-\varepsilon)|t|}{2q}, \qquad \varepsilon > 0, \tag{2}$$

are fulfilled, where $1/q^2 = 1/a^2 + 1/b^2$. Then everywhere in D, $u \le M$.

Proof. Let

$$g(x,y,t) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2.$$

For the function $w = \log u$ we have $\Delta w = -g/u^2$. With the aid of the known inequalities for the moduli of the derivatives of harmonic functions at the center of a ball of fixed radius δ, we obtain $g \le K(\delta)u^2$, $K(\delta) = K > 0$, i.e. $\Delta w \ge -K$, where the constant K depends only on δ. Thus the function $w_1 = w + K(x^2 - a^2)$ is subharmonic in D, and the function $u_1 = ue^{K(x^2-a^2)}$ is logarithmically subharmonic. Condition (1) gives that $u_1 \le M$ on Γ.

Consider the function

$$v(x,y,t) = \exp\left(-\operatorname{ch}\frac{\pi kt}{2q}\cos\frac{\pi kx}{2a}\cos\frac{\pi ky}{2b}\right),$$

$$1 - \varepsilon < k < 1,$$

which is logarithmically subharmonic throughout the cylinder, with $0 < v \leqslant 1$ and

$$\lim_{t\to\infty} u_1[v(x,y,t)]^\sigma = 0$$

for every $\sigma > 0$. As is known (see, for example, ([4]), p. 81), the existence of the indicated function v ensures the applicability of the maximum principle to the logarithmically subharmonic function u_1 in D. Consequently, $u_1 \leqslant M$ in D, and

$$u \leqslant Me^{K(a^2-x^2)} \leqslant Me^{Ka^2}.$$

Since the domain D is contained in the cylinder, one can show that the maximum principle is also applicable in D to the bounded harmonic function u, and therefore $u \leqslant M$.

The theorem remains valid also for harmonic functions in spaces of arbitrary dimension; moreover, if the domain D is contained in the cylinder

$$-a_j \leqslant x_j \leqslant a_j, \quad -\infty < t < \infty; \quad j = 1, \dots, n,$$

then the number $q > 0$ is determined by the relation

$$1/q^2 = \sum_{j=1}^{n} 1/a_j^2.$$

In a subsequent paper the author intends to generalize the method to domains of other types.

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Received
12 XI 1963

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[3] E. L. Solomentsev, Proc. Moscow Power Engineering Inst., issue 42, mathematical (1962).
[4] I. I. Privalov, *Subharmonic Functions*, 1937.

Note: Figure translations are in progress. See original paper for figures.