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Abstract

Full Text

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ON A PROBLEM OF G. FICHERA

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In the works of G. Fichera ([1,2]) a formulation is given of a boundary-value problem for the equation

$$L(u) \equiv a^{ij}u_{x_ix_j} + b^iu_{x_i} + cu = f, \qquad a^{ij}\xi_i\xi_j \geqslant 0, \tag{1}$$

which he calls elliptic-parabolic.* In ([1,2]) a generalized solution of this boundary-value problem is constructed and the question of its uniqueness is posed. In addition, he proves the maximum principle and uniqueness for the classical solution of this problem.

In the present paper we shall prove uniqueness of the generalized solution of G. Fichera' s boundary-value problem, establish for it the maximum principle and, by a method different from that in ([2]), prove the existence of such a solution. By probabilistic methods the degenerate equation (1) was studied in ([3]). Numerous works are devoted to the study of particular classes of equations of the form (1), (see ([4,5]) and others).

We shall assume that equation (1) with the condition $a^{ij}\xi_i\xi_j \geqslant 0$ is given in a bounded domain of the space $(x_1, \dots, x_m)$, $a^{ij} \subset C^{(2+\alpha)}(D)$, $b^i \subset C^{(1+\alpha)}(D)$, $c \subset C^{(\alpha)}(D)$, $0 < \alpha < 1$. Let the domain A be such that $\overline{A} \subset D$ and the boundary Σ of the domain A belongs to the class $A^{(3)}$, (see ([6]), pp. 10–11). Let $n = (n_1, \dots, n_m)$ be the vector of the inward normal to the boundary of the domain. Denote by Σ^0 the set of points of Σ where $a^{ij}n_in_j = 0$. At the points of Σ^0 consider the function $b = (b^i - a^{ij}_{x_j})n_i$. Denote by Σ_1 the set of points of Σ^0 where $b > 0$, by Σ_2 the set of points of Σ^0 where $b < 0$, and by Σ_0 the points of Σ^0 where $b = 0$. The set $\Sigma - \Sigma^0$ will be denoted by Σ_3. By Γ we denote the boundary of the set $\Sigma_0 + \Sigma_2$ on Σ. Let

$$L^*(u) \equiv a^{ij}u_{x_ix_j} + b^{*i}u_{x_i} + c^*u, \tag{2}$$

where $b^{*i} = 2a^{ij}_{x_j} - b^i,\ c^* = a^{ij}_{x_ix_j} - b^i_{x_i} + c$.

Definition. A bounded measurable function u in A will be called a **generalized solution of the boundary-value problem**

$$L(u) = f \text{ in } A, \qquad u = g, \text{ on } \Sigma_2 + \Sigma_3, \tag{3}$$

if for every v in $C^{(2)}(A)$, equal to zero on $\Sigma_1 + \Sigma_3$, the integral identity holds:

$$\int_A uL^*(v)\,dx = \int_A vf\,dx - \int_{\Sigma_3} g\,\partial v/\partial\gamma\,d\sigma + \int_{\Sigma_2} bgv\,d\sigma, \tag{4}$$

where $\partial/\partial\gamma = a^{ij}\cos(n, x_j)\partial/\partial x_i$, $d\sigma$ is the surface-area element of the surface Σ.

Lemma 1. *The sign of the function b at the points Σ^0 does not change under any nondegenerate change of the independent variables in equation (1).*

This assertion is proved by direct verification.

Lemma 2. *Let u satisfy the equation*

$$L_\varepsilon(u) \equiv \varepsilon\Delta u + L(u) = f \text{ in } A, \qquad \varepsilon > 0, \tag{5}$$

$u \subset C^{(1)}(\overline{A})$, $u = 0$ on Σ, $|f| \leqslant M$, $c \leqslant c_0 < 0$. Let the set G on Σ be such that $\overline{G}$ lies inside $\Sigma_0 + \Sigma_2 + \Sigma_3$ and $b < 0$ at the points of G belonging to the boundary of the set Σ_3. Then at the points

$$|u_{x_i}| \leqslant M_1\varepsilon^{-1/2}, \qquad i = 1, \dots, m. \tag{6}$$

* Here, as everywhere below, summation over repeated indices from 1 to m is assumed.

At all points of Σ the inequality

$$|u_{x_i}| \leqslant M_2\varepsilon^{-1}. \tag{7}$$

By M_i and K_i we denote positive constants independent of ε.

Proof. Let $P_0 \subset G$. In a neighborhood of P_0 pass to local coordinates $y_1, \dots, y_m$ with origin at P_0, for which Σ lies in the plane $y_m = 0$. Let the set of points

$$\{\rho^2 = y_1^2 + \dots + y_{m-1}^2 \leqslant 4\delta^2\}$$

be contained in $\Sigma_0 + \Sigma_2 + \Sigma_3$. Put $\psi(\rho) = \sqrt{\varepsilon}$ for $\rho \leqslant \delta$ and

$$\psi(\rho) = \sqrt{\varepsilon}\,[1 - (\rho^2 - \delta^2)^3/27\delta^6]$$

for $\delta \leqslant \rho \leqslant 2\delta$. In the domain

$$\Omega_{\varepsilon\delta}\{0 < \rho < 2\delta,\ 0 < y_m < \psi(\rho)\}$$

consider the function $w = K_0(e^{-z} - 1)$, where

$$z = K_1(y_m + \sqrt{\varepsilon} - \psi)/\sqrt{\varepsilon}.$$

Equation (5) in the variables y has the form

$$L_\varepsilon(u) \equiv \varepsilon\mu^{ij}u_{y_iy_j} + \varepsilon\nu^i u_{y_i} + \alpha^{ij}u_{y_iy_j} + \beta^i u_{y_i} + cu = f.$$

By Lemma 1, the function $(\beta^i - \alpha^{ij}_{y_j})n_i = \beta^m$ at points of Σ^0 has the sign of b. Since $b \leqslant 0$ on $\Sigma_0 + \Sigma_2$, it follows that $\beta^m \leqslant K_2\sqrt{\varepsilon}$ at points of $\Omega_{\varepsilon\delta}$ whose distance from $\Sigma_0 + \Sigma_2$ is not greater than $\sqrt{\varepsilon}$. Since, moreover, $b < 0$ at boundary points Σ_3 on G, $\alpha^{mm} > 0$ on Σ_3, and

$$|\psi| \leqslant \sqrt{\varepsilon}, \qquad |\psi_{y_i}| \leqslant K_3\sqrt{\varepsilon}, \qquad |\psi_{y_iy_j}| \leqslant K_3\sqrt{\varepsilon},$$

it is easy to see that

$$L_\varepsilon(w) \geqslant |c_0|K_0$$

in $\Omega_{\varepsilon\delta}$ for sufficiently large K_1 and small ε.

Choosing K_0 sufficiently large, we obtain that $L_\varepsilon(w \pm u) \geqslant 0$ in $\Omega_{\varepsilon\delta}$ and $w \pm u \leqslant 0$ on the boundary of $\Omega_{\varepsilon\delta}$. Consequently, $w \pm u \leqslant 0$ in $\Omega_{\varepsilon\delta}$. Since $w = u = 0$ on Σ for $\rho \leqslant \delta$, at these points

$$|u_{y_m}| \leqslant K_0K_1\varepsilon^{-1/2}.$$

This is sufficient for the proof of (6) on G.

The estimate (7) on Σ is obtained analogously, putting $\psi(\rho) = \varepsilon$ for $\rho \leqslant \delta$,

$$\psi(\rho) = \varepsilon[1 - (\rho^2 - \delta^2)^3]/27\delta^6$$

for $\delta \leqslant \rho \leqslant 2\delta$, and

$$z = K_1(y_m + \varepsilon - \psi)/\varepsilon$$

for the function w.

Lemma 3. Let v satisfy the equation

$$\varepsilon\Delta v + L^*(v) = \Phi, \qquad \varepsilon > 0, \tag{8}$$

in some domain A^* with boundary Σ^* of class $A^{(3)}$. Suppose that $v \subset C^{(2)}(\bar{A}^*)$, $v = 0$ on Σ^*. Let Σ_3^* denote the set of points of Σ^* where

$$a^{ij}n_in_j \neq 0.$$

We shall assume that

$$b^* = (b^{*i} - a^{ij}_{x_j})n_i \leqslant 0$$

on $\Sigma^* - \Sigma_3^*$, $b^* < 0$ on the boundary of the set Σ_2^*;

$$|\Phi| \leqslant M, \qquad c^* \leqslant c_1 < 0, \qquad c - b^*_{x_i}/2 \leqslant c_2 < 0.$$

Then

$$\varepsilon^2 \int_{A^*} (\Delta v)^2\, dx \leqslant M_3. \tag{9}$$

Proof. Multiply (8) by v and integrate over A^*. Integrating by parts, we obtain

$$\int_{A^*} \varepsilon v_{x_i} v_{x_i}\, dx + \int_{A^*} a^{ij} v_{x_i} v_{x_j}\, dx + \int_{A^*} v^2\, dx \leqslant M_4 \int_{A^*} \Phi^2\, dx. \tag{10}$$

Next, multiply (8) by $\varepsilon \Delta v$ and integrate over A^*. We have

$$\int_{A^*} \varepsilon^2 (\Delta v)^2\, dx + \int_{A^*} \varepsilon \Delta v\, a^{ij} v_{x_i x_j}\, dx = \int_{A^*} (\Phi - c^* v - b^{*i} v_{x_i}) \varepsilon \Delta v\, dx. \tag{11}$$

By Lemma 2 and the assumptions concerning Σ^*, at all points of Σ^* the estimate (6) holds for v. Taking into account (10) and (6), and integrating by parts, we obtain

$$\left| \int_{A^*} \varepsilon \Delta v\, (\Phi - c^* v)\, dx \right| \leqslant \frac{1}{2} \int_{A^*} \varepsilon^2 (\Delta v)^2\, dx + \frac{1}{2} \int_{A^*} (\Phi - c^* v)^2\, dx;$$

$$\left| \int_{A^*} \varepsilon b^{*i} v_{x_i} v_{x_j x_j}\, dx \right| = \left| \int_{A^*} \left(\varepsilon b^{*i}_{x_i} v_{x_j} v_{x_j}/2 - \varepsilon b^{*i}_{x_j} v_{x_i} v_{x_j} \right) dx + \int_{\Sigma^*} \varepsilon b^{*i} \left(v_{x_i} v_{x_j} \cos(n, x_j) - \frac{1}{2} v_{x_j} v_{x_j} \cos(n, x_i) \right) d\sigma \right| \tag{12}$$

$$\int_{A^*} \varepsilon v_{x_k x_k} a^{ij} v_{x_i x_j}\, dx = \int_{A^*} \varepsilon a^{ij} v_{x_k x_i} v_{x_k x_j}\, dx + \int_{A^*} \varepsilon \left[a^{ij}_{x_j} v_{x_k} v_{x_i x_k} - a^{ij}_{x_k} v_{x_k} v_{x_i x_j} \right] dx +$$

$$+ \int_{\Sigma^*} \varepsilon a^{ij} v_{x_k} \left[v_{x_i x_j} \cos(n, x_k) - v_{x_i x_k} \cos(n, x_j) \right] d\sigma = I_1 + I_2 + I_3. \tag{13}$$

We note that $I_1 \geq 0$, while I_2 can be transformed by integration by parts analogously to (12), and one can prove that $|I_2| \leq M_6$. To estimate I_3, we divide Σ^* into pieces Σ^{*l} $(l = 1, \dots, N)$ and introduce, in a neighborhood of Σ^{*l}, local coordinates $y^i = y^i(x_1, \dots, x_m)$, chosen so that $y^m = 0$ contains Σ^{*l}. Since $v = 0$ on Σ^*, it follows that

$$I_3 = \int_{\cup \Sigma^{*l}} \frac{1}{2} \varepsilon a^{ij} (v^2_{y^m})_{y^s} \left[y^m_{x_k} y^m_{x_k} y^s_{x_i} y^m_{x_j} - y^m_{x_k} y^s_{x_k} y^m_{x_i} y^m_{x_j} \right] \chi(y)\, dy' +$$

$$+ \int_{\Sigma^{*l}} \varepsilon a^{ij} v^2_{y^m} \left(y^m_{x_k} y^m_{x_k} y^m_{x_i x_j} - y^m_{x_k} y^m_{x_i} y^m_{x_i x_j} \right) \chi\, dy',$$

where $dy' = dy^1 \cdots dy^{m-1}$, χ depends only on y^i, $s = 1, \dots, m-1$. Integrating by parts in the first integral entering I_3, we obtain $|I_3| \leq M_7$. The lemma is proved.

Theorem 1. Let $c \leq c_0 < 0$; let f and g be bounded and measurable functions on A and $\Sigma_2 + \Sigma_3$, respectively; and let Γ have measure zero on Σ. Then in

A there exists a generalized solution of the boundary-value problem (3), which satisfies the inequality (maximum principle)

$$|u| \leq \max\{\sup|f|/|c_0|,\ \sup|g|\}. \tag{14}$$

Proof. Let $f_n \subset C^{(\infty)}(\overline{A})$, $f_n \to f$ as $n \to \infty$ in the norm $\mathcal{L}_2(A)$, and let $|f_n| \leq \sup|f|$; and let $g_n \subset C^{(\infty)}(\Sigma_2+\Sigma_3)$, $g_n \to g$ in the norm $\mathcal{L}_2(\Sigma_2+\Sigma_3)$ as $n \to \infty$. Let $\tilde{g}_n \subset C^{(2)}(\overline{A})$, $\tilde{g}_n = g_n$ on $\Sigma_2+\Sigma_3$, and $|\tilde{g}_n| \leq \sup|g|$. Let $u_{\varepsilon n}$ be the solution of the problem $L_\varepsilon(u) = f_n$ in A and $u = \tilde{g}_n$ on Σ. By the maximum principle, (14) is valid for $u_{\varepsilon n}$. Since for $z_{\varepsilon n} = u_{\varepsilon n} - \tilde{g}_n$ we have $L_\varepsilon(z_{\varepsilon n}) = f_n - L_\varepsilon(\tilde{g}_n)$, Lemma 2 is valid for $u_{\varepsilon n}$ with fixed n. Let $v \subset C^{(2)}(\overline{A})$ and $v = 0$ on $\Sigma_1+\Sigma_3$. Applying Green' s formula, we obtain

$$\int_A f_n v\,dx = \int_A \varepsilon\Delta v\, u_{\varepsilon n}\,dx + \int_A L^*(v)\,u_{\varepsilon n}\,dx +$$
$$+\varepsilon\int_\Sigma u_{\varepsilon n}\frac{\partial v}{\partial n}\,d\sigma + \int_{\Sigma_3} g_n\frac{\partial v}{\partial\gamma}\,d\sigma - \int_{\Sigma_2} bg_n v\,d\sigma - \int_{\Sigma_0+\Sigma_2}\varepsilon v\frac{\partial u_{\varepsilon n}}{\partial n}\,d\sigma. \tag{15}$$

Let $u_{\varepsilon_k n}$ converge weakly to u_n as $\varepsilon_k \to 0$. Passing to the limit in (15) as $\varepsilon_k \to 0$, we obtain that u_n satisfies (4) with f_n and g_n, since the last integral in (15) in a δ-neighborhood Σ^δ of the set Γ can be estimated according to (7), while on $\Sigma_0+\Sigma_2-\Sigma^\delta$, according to (6), and, evidently, it is arbitrarily small for sufficiently small δ and ε. Let $u_{n_k} \to u$ weakly as $n_k \to \infty$. Then, passing to the limit in (4) for u_{n_k} as $n_k \to \infty$, we obtain the assertion of the theorem.

Theorem 2. Let $c^* \leq c_1 < 0$, $2c - b^i_{x_i} \leq 2c_2 < 0$, and $\Gamma \subset A^{(2)}$. Let the function u from $\mathcal{L}_p(A)$, with $p \geq 6$, be such that

$$\int_A L^*(v)u\,dx = 0 \tag{16}$$

for every v from $C^{(2)}(\overline{A})$ equal to zero on $\Sigma_3+\Sigma_1$. Then $u = 0$ almost everywhere in A.

Proof. Let the domain Ω be such that $D \supset \Omega \supset A+\Sigma_2+\Sigma_0+\Omega_\delta$, the boundary S of the domain Ω belongs to the class $A^{(3)}$, $S \supset \Sigma_1+\Sigma_3-(\Sigma\cap\Omega_\delta)$, where Ω_δ is some domain containing the δ-neighborhood of Γ

and $\Omega_\delta \to \Gamma$ as $\delta \to 0$. Let $a(x) \subset C^{(2+\alpha)}(\Omega)$, $a = 0$ in $\bar{A}$, $a > 0$ on $\Omega - \bar{A}$. Let v satisfy in Ω the equation

$$\varepsilon\Delta\bar{v} + L^*(\bar{v}) + a\Delta\bar{v} = \Phi \tag{17}$$

and the condition $\bar{v} = 0$ on S, where Φ is a smooth function, finite in A. Obviously, at points of S either $a^{ij}n_in_j + an_in_i \neq 0$, or $b^* = (b^i - a^{ij}_{x_j})n_i < 0$.

Therefore (9) is valid for $\bar{v}$. Let φ^δ be a function such that $\varphi^\delta \subset C^{(\infty)}(\Omega)$, $\varphi^\delta = 0$ in Ω_δ and $\varphi^\delta = 1$ in any closed domain lying in A, if δ is sufficiently small, $0 \le \varphi^\delta \le 1$.

The function $v = \bar{v}\varphi^\delta$ can be substituted into (16), since $v = 0$ on $\Sigma_1 + \Sigma_3$. For sufficiently small δ we have

$$\int_A \Phi u\,dx = \int_A \varepsilon\Delta\bar{v}\,\varphi^\delta u\,dx - \int_A \left(L^*(\varphi^\delta) - c^*\varphi^\delta\right)\bar{v}u\,dx - 2\int_A a^{ij} v_{x_i}\varphi^\delta_{x_j} u\,dx. \quad (18)$$

We shall show that the right-hand side of (18) is arbitrarily small for sufficiently small δ and $\varepsilon(\delta)$, i.e., the left-hand side of (18) is equal to zero and $u = 0$ almost everywhere. Let $u_n \subset C^{(\infty)}(\bar{A})$, be finite in A, and $u_n \to u$ in $\mathcal{L}_2(A)$ as $n \to \infty$. By (9) we have

$$\left|\int_A \varepsilon\Delta\bar{v}\,\varphi^\delta u\,dx\right| \le \left(\int_A \varepsilon^2(\Delta\bar{v})^2\,dx\right)^{1/2}\left(\int_A (u-u_n)^2\,dx\right)^{1/2} = \left|\int_A \varepsilon\bar{v}\Delta(\varphi^\delta u_n)\,dx\right| \to 0$$

as $n \to \infty$, $\varepsilon(n) \to 0$, and fixed δ. To estimate the two last integrals in (18), divide a neighborhood of Γ into pieces ω_l so that in ω_l one can introduce local coordinates y_i such that $\Sigma \cap \omega_l$ lies on $y_m = 0$ and $|y_m| \le \delta$, $|y_{m1}| < \delta$ at points of Ω_δ, and for $\omega_k \cap \omega_l$ we have $y^k_m = y^l_m$, $y^k_{m-1} = y^l_{m-1}$. Now take the function $\varphi^\delta = \varphi((\delta y^2_{m-1} + y^2_m)/\delta^2)$, where $\varphi(s) = 0$ for $s \le 1$, $\varphi(s) = 1$ for $s \ge 2$, $0 \le \varphi \le 1$, and $\varphi(s)$ is a smooth function of s. We have

$$\int_{A\cap\omega_k} \left(L^*(\varphi^\delta) - c^*\varphi^\delta\right)\bar{v}u\,dx = \int_{A\cup\omega_k} \left(\alpha^{ij}\varphi^\delta_{y_i y_j} + \beta^i\varphi^\delta_{y_i}\right)\bar{v}u\chi\,dy, \quad (19)$$

where χ depends only on y_i. Taking into account that the domain where $\varphi^\delta \ne 0$, $\varphi^\delta_{y_i} \ne 0$, $\varphi^\delta_{y_i y_j} \ne 0$, has order $\delta^{3/2}$, and that in this domain

$$\varphi^\delta_{y_m} = O(\delta^{-1}), \qquad \varphi^\delta_{y_{m-1}} = O(\delta^{-1/2}), \qquad \varphi^\delta_{y_m y m} = O(\delta^{-2}), \qquad \varphi^\delta_{y_{m-1}y_{m-1}} = O(\delta^{-1}),$$

$$\varphi^\delta_{y_{m-1}y_m} = O(\delta^{-3/2}), \qquad \alpha^{mm} = O(\delta), \qquad \alpha^{mm-1} = O(\delta^{1/2}), \qquad \alpha^{m-1,m-1} = O(1), \quad (20)$$

$$|\chi\bar{v}| \le M_8, \qquad u \subset \mathcal{L}_\theta(A),$$

applying Hölder' s inequality, we obtain that the integral (19) tends to zero as $\delta \to 0$. We estimate the last integral in (18), using (10) for $\bar{v}$:

$$\left|\int_{A\cap\omega_k} a^{ij} v_{x_i} \varphi^\delta_{x_j} u\,dx\right| \le \left(\int_{A\cap\omega_k} a^{ij} v_{x_i} v_{x_j}\,dx\right)^{1/2} \left(\int_{A\cap\omega_k} a^{ij} \varphi^\delta_{x_i} \varphi^\delta_{x_j} u^2\,dx\right)^{1/2}. \tag{21}$$

To estimate the last integral in (21), as in (19), we pass to the coordinates y and use the estimates (20). The theorem is proved.

Remark. If Γ is the empty set or $b = 0$ on Γ, then the proof of Theorem 2 remains valid for $u \subset \mathcal{L}_2(A)$. The smoothness assumptions on Σ can be weakened. Thus, for example, the proof of Theorem 1 remains valid if Σ_1 is assumed only piecewise smooth, and in Theorem 2 one may assume that $\Sigma_2+\Sigma_0$ is piecewise smooth.

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Note: Figure translations are in progress. See original paper for figures.