# I. S. Shikin

1964

SovietRxiv

## Abstract

## Full Text

## I. S. Shikin

# RIEMANN WAVES IN RELATIVISTIC MAGNETIC HYDRODYNAMICS

*(Presented by Academician L. I. Sedov on 19 VI 1964)*

The paper considers the general equations of relativistic magnetic hydrodynamics (M.H.D.) for a nonviscous, non-heat-conducting, and perfectly conducting substance, and Riemann waves (simple waves). Previously, simple waves in relativistic M.H.D. were considered in ($^4$).

1. The equations of relativistic M.H.D. for an ideal substance are:

the laws of conservation of momentum—energy

$$\partial\left(T_{ik}^{\text{subst}}+T_{ik}^{\text{field}}\right)/\partial x_k=0, \tag{1}$$

where the energy—momentum tensor of an ideal fluid is

$$T_{ik}^{\text{subst}}=wu_iu_k+p\delta_{ik}, \tag{2}$$

$T_{ik}^{\text{field}}$ is the energy—momentum tensor of the electromagnetic field (the notation coincides with the notation of ($^1$); $i,k,l,m=1,2,3,4,\ \alpha=1,2,3;\ x_4=ict$);

the induction equation

$$\operatorname{rot}E=-\partial H/c\,\partial t; \tag{3}$$

the condition of ideal conductivity

$$\mathbf{E}=-[\mathbf{v},\mathbf{H}]/c. \tag{4}$$

With the aid of the electromagnetic-field tensor $F_{ik}$, condition (4) is written in the form

$$F_{ik}u_k=0. \tag{4'}$$

In expression (2), the heat function $w$, according to the equation of state, is, generally speaking, a function of the pressure $p$ and of the number of particles

$n$ per unit proper volume. Depending on the form of the equation of state there may be two cases:

**Case I:** $w = w(p, n)$. In this case the system of relations (1)–(4) is not closed. To close it we shall use the law of conservation of the number of particles

$$\partial(nu_k)/\partial x_k = 0. \tag{5}$$

**Case II:** $w = w(p)$. In this case we shall use only the relations (1)–(4), which by themselves form a closed system. Such a case is realized, for example, in the extreme relativistic limit for temperatures, when $w = 4p$.

Equations (1) can be transformed by multiplying by $u_i$ and by $u_i u_l + \delta_{il}$. Using (4′), and in case I also using (5), we obtain

$$\begin{array}{ll} \text{in case I} & \text{in case II} \\ \dfrac{d}{ds}\left(\dfrac{w}{n}\right) - \dfrac{1}{n}\dfrac{dp}{ds} = 0, \quad \dfrac{d}{ds} \equiv u_i \dfrac{\partial}{\partial x_i}; & \dfrac{\partial}{\partial x_k}\left[\dfrac{w(p)}{B(p)} u_k\right] = 0, \quad B(p) = \exp \displaystyle\int \dfrac{dp}{w(p)}; \\ n\dfrac{d}{ds}\left(\dfrac{w}{n} u_l\right) = -\dfrac{\partial p}{\partial x_l} + \dfrac{1}{4\pi} F_{lk} \dfrac{\partial F_{km}}{\partial x_m}; & \dfrac{wu_k}{B}\dfrac{\partial}{\partial x_k}\left[Bu_l\right] = -\dfrac{\partial p}{\partial x_l} + \dfrac{1}{4\pi} F_{lk} \dfrac{\partial F_{km}}{\partial x_m}. \end{array} \tag{6}$$

If in case II the chemical potential is equal to 0, then the scalar equation in (6) is the continuity equation for the entropy. In each of the systems (6) only 4 relations are independent. As is seen from (6), the direct use of the 4-velocity $u_i$ ($u_i^2 = -1$), $w$, and $n$ proves inconvenient. Instead of them we shall use the following quantities, calling them variables of an auxiliary gas ([2], [3]):

$$\text{in case I} \quad \tilde{v}_i = (w/mnc^2)cu_i, \quad \tilde{\rho} = (mnc^2/w)mn, \quad \tilde{I} = (w/mnc^2)^2 c^2/2; \tag{7}$$

$$\text{in case II} \quad \tilde{v}_i = B(p)u_i, \quad \tilde{\rho} = w(p)/B^2(p), \quad \tilde{I} = B^2(p)/2. \tag{8}$$

We shall also use the following expressions for the square of the sound speed of the auxiliary gas (2):

in case I

$$\tilde{a}^2 = (w/mnc^2)^2 \omega^2/[1 - (\omega/c)^2]; \tag{9}$$

in case II

$$\tilde{a}^2 = B^2(p)\,\omega^2/[1 - (\omega/c)^2], \tag{10}$$

where $\omega$ is the relativistic sound speed, equal in case I to $\omega = c\sqrt{(\partial p/\partial e)_\sigma/n}$ ($e = w - p$ is the internal energy, $\sigma$ is the entropy per unit proper volume), and in case II to $\omega = c\sqrt{dp/de}$.

In the nonrelativistic limit, in case I one must consider only the spatial components of the 4-vector equation (6), and in the expression for the ponderomotive force one should omit the terms containing the electric field. In the nonrelativistic case it is also assumed that the magnetic field (and the current density) is not transformed in passing from one inertial system to another, so that the equations of nonrelativistic MHD are Galilean invariant. The quantities $\mathfrak{v}$, $\rho$, and $\tilde{I}$ then have the meaning of 3-velocity, density, and heat function per unit mass.

2. A Riemann wave (simple wave) is a one-dimensional nonstationary motion in which the wave parameters in some laboratory system depend on the coordinate $x$ along the direction of propagation of the wave and on the time $t$ through some combination $\varphi(x,t)$.

We shall consider both cases I and II in parallel. To simplify the notation we shall omit the tilde over the quantities (7)–(10), noting when necessary the difference between the results for these two cases.

The system of basic equations for a simple wave in the variables of the auxiliary gas is written in the form ($v_0 = v_4/i$):

$$d(\rho v_x) - \beta d(\rho v_0) = 0, \tag{11}$$

$$4\pi\rho\,(v_x - \beta v_0)\,dv_x = -dp - d(H_y^2 + H_z^2 + E_y^2 + E_z^2 - E_x^2)/2 + \beta d(E_y H_z - E_z H_y), \tag{12}$$

$$4\pi\rho\,(v_x - \beta v_0)\,dv_y = d(H_x H_y + E_x E_y) + \beta d(E_z H_x - E_x H_z), \tag{13}$$

$$4\pi\rho\,(v_x - \beta v_0)\,dv_z = d(H_x H_z + E_x E_z) + \beta d(E_x H_y - E_y H_x), \tag{14}$$

$$dE_z + \beta\, dH_y = 0, \qquad dE_y - \beta\, dH_z = 0, \tag{15}$$

$$H_x = \text{const}, \qquad E_x v_0 = H_y v_z - H_z v_y, \qquad E_y v_0 = H_z v_x - H_x v_z, \tag{16}$$
$$E_z v_0 = H_x v_y - H_y v_x.$$

In case I, to relations (11)–(16) one must add the equation

$$(v_x - \beta v_0)(\rho\, dI - dp) = 0. \tag{17}$$

The quantity $c\beta \equiv V = -\varphi_t/\varphi_x = \partial x(t,\varphi)/\partial t$ is the propagation speed of the points of the front of constant phase $\varphi(x,t) = \text{const}$. It also depends on $\varphi$.

In equations (11)–(16) there appear quantities measured in the laboratory coordinate system under consideration. Instead of them we shall use quantities measured, for each phase $\varphi(x,t) = \text{const}$, in that coordinate system (moving with speed $V$ along $x$) in which this phase is at rest ( “frozen” ). We shall mark these quantities with a prime. The connection between the primed and unprimed quantities is given by the Lorentz formulas:

$$\begin{gathered}
v_x = v'_x \operatorname{ch}\xi + v'_0 \operatorname{sh}\xi, \qquad v_y = v'_y, \qquad v_z = v'_z, \qquad v_0 = v'_0 \operatorname{ch}\xi + v'_x \operatorname{sh}\xi;\\
H_x = H'_x, \qquad H_y = H'_y \operatorname{ch}\xi - E'_z \operatorname{sh}\xi, \qquad H_z = H'_z \operatorname{ch}\xi + E'_y \operatorname{sh}\xi,\\
E_x = E'_x, \qquad E_y = E'_y \operatorname{ch}\xi + H'_z \operatorname{sh}\xi, \qquad E_z = E'_z \operatorname{ch}\xi - H'_y \operatorname{sh}\xi;\\
\operatorname{ch}\xi = \frac{1}{\sqrt{1-(V/c)^2}}, \qquad \operatorname{sh}\xi = \frac{V/c}{\sqrt{1-(V/c)^2}}, \qquad \operatorname{th}\xi = V/c.
\end{gathered} \tag{18}$$

With the aid of (18), relations (11), (13)–(15) are brought to the form

$$d(\rho v'_x) + \rho v'_0\, d\xi = 0, \tag{19}$$

$$4\pi\rho v'_x\, dv'_y = H_x\, dH'_y - H_x E'_z\, d\xi + E'_y\, dE_x, \tag{20}$$

$$4\pi\rho v'_x\, dv'_z = H_x\, dH'_z + H_x E'_y\, d\xi + E'_z\, dE_x, \tag{21}$$

$$dE'_z = H'_y\, d\xi, \qquad dE'_y = -H'_z\, d\xi. \tag{22}$$

Equation (12) and the relation

$$\rho\left(v_0\frac{\partial v_0}{\partial x_0} + v_x\frac{\partial v_0}{\partial x}\right) = \frac{\partial p}{\partial x_0} - \frac{1}{4\pi}\times$$

$$\times\frac{\partial}{\partial x}(E_yH_z - E_zH_y) - \frac{1}{8\pi}\frac{\partial}{\partial x_0}(E^2+H^2), \quad x_0 = ct$$

(the time component in system (6)), with the aid of (22), are transformed to the form

$$8\pi\rho v'_x(dv'_x + v'_0\, d\xi) = -8\pi\, dp - d(H^{*2}),$$

$$H^{*2} \equiv H_x^2 + H_{y'}^2 + H_{z'}^2 - E_x^2 - E_{y'}^2 - E_{z'}^2, \tag{23}$$

where $H^{*2}$ is the square of the magnitude of the magnetic field in the proper reference frame of the matter element,

$$4\pi\rho v'_x(dv'_0 + v'_x\, d\xi) = E'_z\, dH'_y - E'_y\, dH'_z - (E_{y'}^2 + E_{z'}^2)\, d\xi. \tag{24}$$

With the aid of (19), from (23) we obtain

$$d(H^{*2}) = 8\pi(v_{x'}^2\, d\rho - dp). \tag{25}$$

Differentiating the expressions

$$E'_y v'_0 = H'_z v'_x - H_x v'_z \quad \text{and} \quad E'_z v'_0 = H_x v'_y - H'_y v'_x$$

(the condition of ideal conductivity), with the aid of (20)–(21), (22), (24), and (19), we obtain

$$v_{x'}^2 H'_z\, d\rho = \rho(v_{x'}^2 - a_A^2)(E'_y\, d\xi + dH'_z), \tag{26}$$

$$v_{x'}^2 H'_y\, d\rho = \rho(v_{x'}^2 - a_A^2)(-E'_z\, d\xi + dH'_y), \tag{27}$$

where

$$a_A^2 = (H_\parallel^2 - E_\perp'^2)/4\pi\rho, \qquad H_\parallel \equiv H_x, \qquad E_\perp'^2 \equiv E_{y'}^2 + E_{z'}^2. \tag{28}$$

From (26)–(27), taking into account that

$$H_x E_x + H'_y E'_y + H'_z E'_z = 0,$$

we obtain

$$(v_{x'}^2 - a_A^2)(H_x E_x\, d\xi + H'_x\, dH'_y - H_y\, dH'_z) = 0. \tag{29}$$

As the basic system we shall use relations (19), (22), (25), (26), (27), and (17).

First let us consider the case when the wave does not propagate with respect to the particles ($v'_x = 0$). In such a wave $d\xi/d\rho = 0$, $V = \text{const}$. The wave propagates without distortion of its profile. If, moreover, $H_\parallel = 0$, then $E'_y = E'_z = 0$. The quantities $v_y$ and $v_z$ may vary arbitrarily, while the remaining quantities are connected by the relation

$$p + [(H_\perp'^2 - E_\parallel^2)/8\pi] = \text{const}, \quad E_\parallel \equiv E_x, \quad H_\perp'^2 \equiv H_{y'}^2 + H_{z'}^2.$$

Such a wave is called a **tangential** wave. In the general case the front of a tangential wave has arbitrary width. In the particular case when the width of the wave region is zero, the wave is a **tangential discontinuity**.

If, in a simple wave, $v'_x = 0$, but $H_x \neq 0$, then $\mathbf{v} = \text{const}$, $\mathbf{H} = \text{const}$, $\mathbf{E} = \text{const}$, $p = \text{const}$. In case II also $\rho = \text{const}$, so that all quantities are constant, and we have simply a region of uniform motion. In case I, however, $\rho \equiv (mnc)^2/w$ may vary in the wave in an arbitrary manner, and together with it all the other thermodynamic quantities, except for the pressure $p$ (in particular, the entropy).

The wave in this case is called an **entropy** wave. A particular case of such a wave is a **contact discontinuity**.

From equation (29) it is seen that two possibilities may occur.

First, $v_{x'}^2 = a_A^2$. In this case $\rho = \text{const}$. It also follows from the equations that $d\xi/d\rho = 0$, i.e. the wave propagates without distortion of its profile, $v_x' = \text{const}$, $E_y' = \text{const}$, $E_z' = \text{const}$, $H^{*2} = \text{const}$. In the original relativistic variables we have:

$$u_x^2 = (H_\parallel^2 - E_\perp'^2)/4\pi w. \tag{30}$$

Such a wave is called an **Alfvén wave**. Among the $v$-systems in which the front of the Alfvén wave is at rest, one can choose one in which $E' = 0$. In this system we have $4\pi\rho v_{x'}^2 = H_x H_\alpha'$, and the velocity is parallel to the magnetic field. In such a system ${H'}^2$ is constant in the wave, while the orientation of the transverse magnetic field $\mathbf{H}_\perp'$ is a parameter that may vary arbitrarily. With respect to the gas at rest in front of it, with magnetic field $\mathbf{H}^*$, an Alfvén wave propagates along the normal to the front with velocity $D_A$:

$$D_A^2 = c^2 H_x^{*2}/(4\pi w + H^{*2}). \tag{31}$$

The second possibility in equation (29) consists in setting the second bracket equal to zero; this corresponds to **magnetosonic** simple waves. In a magnetosonic wave the quantity $d\xi/d\varphi \neq 0$, so that the wave profile is distorted as it propagates. Points of fronts of constant phase move according to the law $x = V(\varphi)t + F(\varphi)$. If $F(\varphi) = 0$, the wave is centered. The quantity $\rho$ (and together with it $p$, which is a function of $\rho$) is variable in the wave, and it may be taken as the variable parameter of the wave. From the system (26), (27) we obtain

$$\rho d(H_\perp'^2 - E_\perp'^2)/d\rho = 2v_{x'}^2 H_\perp'^2/(v_{x'}^2 - a^2\Lambda). \tag{32}$$

On the other hand, differentiating the relation $H_x E_x + H_y' E_y' + H_z' E_z' = 0$ (the condition of orthogonality of $\mathbf{E}'$ and $\mathbf{H}'$) and using (26), (27), we obtain

$$\rho d(E_x^2)/d\rho = 2v_{x'}^2 E_x^2/(v_{x'}^2 - a^2\Lambda). \tag{33}$$

Comparing (32), (33) with (25), taking into account that $dp = a^2 d\rho$, we arrive at the biquadratic equation for $v_x$:

$$v_{x'}^4 - v_{x'}^2 \left[a^2 + (H^{*2}/4\pi\rho)\right] + a^2 \left[(H_\parallel^2 - E_\perp'^2)/4\pi\rho\right] = 0. \tag{34}$$

Equation (34) coincides with the equation for the velocities of small perturbations and physically expresses the fact that points of a front of constant phase propagate through the matter with the velocity of a magnetosonic wave of small amplitude. In the $(x, t)$ plane, the lines of constant phase $\varphi = \text{const}$, forming a pencil of straight lines, coincide with the characteristics of the corresponding family.

In the original relativistic quantities, equation (34) takes the form

$$u_{x'}^4 - u_{x'}^2 \left( \frac{\omega^2}{c^2 - \omega^2} + \frac{H^{*2}}{4\pi w} \right) + \frac{H_\parallel^2 - E_\perp'^2}{4\pi w} \frac{\omega^2}{c^2 - \omega^2} = 0. \tag{35}$$

Let us introduce the angle $\alpha$ of the vector $\mathbf{H}'_\perp$, and the angle $\beta$ of the vector $\mathbf{E}'_\perp$, with the $y$-axis. Setting the second bracket in (29) equal to zero leads to the equation

$$d\alpha/d\xi = H_x E_x / H_\perp'^2. \tag{36}$$

From equations (22) we obtain

$$d\beta/d\xi = -H_x E_x / E_\perp'^2. \tag{37}$$

As is seen from (33), a solution exists when $E_x = 0$ throughout the whole region of the wave. Such a case is realized, in particular, if the matter in front of the wave is at rest. In this case, according to (36), (37), the orientation of the orthogonal vectors $\mathbf{H}'_\perp$ and $\mathbf{E}'_\perp$ remains unchanged in the wave. If, however, $E_x \neq 0$, then the vectors $\mathbf{H}'_\perp$ and $\mathbf{E}'_\perp$ rotate in the wave. Rotation of the vector $\mathbf{H}'_\perp$ is absent in the nonrelativistic theory and is a purely relativistic effect. The angle $\alpha - \beta$ between $\mathbf{H}'_\perp$ and $\mathbf{E}'_\perp$ increases or decreases monotonically (with respect to $\xi$) in the wave, depending on the sign of $H_x E_x$. In both cases the vectors $\mathbf{H}'_\perp$ and $\mathbf{E}'_\perp$ tend to become orthogonal.

Scientific Research Institute of Mechanics
Moscow State University
named after M. V. Lomonosov

Received
1 VI 1964

## References

1. L. D. Landau, E. M. Lifshitz, *Field Theory*, Moscow, 1962.
2. I. S. Shikin, DAN, **142**, 296 (1962).

3. I. S. Shikin, DAN, **142**, 564 (1962).

4. I. A. Akhiezer, R. V. Polovin, ZhETF, **36**, 1845 (1959).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*