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Abstract

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E. I. ZVEROVICH

BOUNDARY-VALUE PROBLEMS WITH SHIFT ON ABSTRACT RIEMANN SURFACES

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1. Let R be a closed Riemann surface of genus p. Let D^+ be a domain on R, bounded by a smooth* contour L, consisting of $m+1$ nonintersecting Jordan curves $L_0, L_1, \dots, L_m$.

Let h $(0 \leq h \leq p)$ be the genus of the surface D^+. Denote by D^- the complement of $\overline{D^+}$ in R. Let $Q_1 = \alpha_\pm(Q)$ be a homeomorphism of the contour L onto itself, where α_+ preserves, and α_- reverses, the orientation of the curves of the contour L. Suppose that the derivative of the function $\alpha_\pm$ with respect to the local parameter does not vanish and is H-continuous. On L two H-continuous functions $G(Q) \neq 0$ and $g(Q)$ are given.

The following boundary-value problems are posed:

Find a piecewise-analytic function $\varphi^\pm(P)$ on R, H-continuously extendable to L, satisfying one of the following boundary conditions:

$$\varphi^+[\alpha_+(Q)] = G(Q)\varphi^-(Q) + g(Q) \quad \text{on } L \tag{1}$$

(Gaseman problem);

$$\varphi^+(Q) = G(Q)\overline{\varphi^-[\alpha_-(Q)]} + g(Q) \quad \text{on } L \tag{2}$$

(a problem of Gaseman-problem type).

Find two functions $\varphi(P)$, $\psi(P)$, analytic in D^+, H-continuously extendable to $\overline{D^+}$, satisfying one of the conditions on L:

$$\varphi[\alpha_+(Q)] = G(Q)\overline{\psi(Q)} + g(Q), \tag{3}$$

$$\varphi[\alpha_-(Q)] = G(Q)\psi(Q) + g(Q). \tag{4}$$

Assuming that $\alpha_\pm[\alpha_\pm(Q)] \equiv Q$ (Carleman condition), find a function $\varphi(P)$, analytic in D^+ and H-continuous in $\overline{D^+}$, satisfying one of the conditions on L:

$$\varphi[\alpha_+(Q)] = G(Q)\overline{\varphi(Q)} + g(Q) \tag{5}$$

(a problem of Carleman-problem type);

$$\varphi[\alpha_-(Q)] = G(Q)\varphi(Q) + g(Q) \tag{6}$$

(Carleman problem).

Problem (1), when $\alpha_+(Q) \equiv Q$, becomes the Riemann problem on L, studied by Yu. L. Rodin ([1,2]), V. Koppelman ([3]), and other authors. Problem (5) is a generalization of the Hilbert problem, well studied by various authors: D. A. Kveselava, I. N. Vekua, B. V. Boyarskii (an account of the results is available in ([4,5])), Yu. L. Rodin ([2]), and other authors. Problems (1)–(6) have so far been studied only in the plane, in the works of D. A. Kveselava ([6]), I. B. Simonenko ([7]), G. S. Litvinchuk ([8]), and E. G. Khasabov ([9]).

In the present note a method is proposed which makes it possible to reduce problems (1)–(4) to the Riemann problem on a certain Riemann surface R', constructed from the original surface R or D^+. This makes it possible to transfer to problems (1)–(5) all qualitative results known for the Riemann problem. In addition, a previously unknown exact upper estimate has been obtained for the number of solutions of the Riemann problem in a special case, which also carries over to problems (1)–(5). Some qualitative results have been obtained for problem (6).

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* Authors who previously studied problems (2)–(6) assumed that L is a Lyapunov contour.

2. Suppose in problem (1) that $g(Q) \equiv 0$; $\alpha_+(Q) \equiv Q$ (the homogeneous Riemann problem). Let

$$\varkappa = \frac{1}{2\pi} \arg G(Q) \Big|_L .$$

Theorem 1. *For $0 \le \varkappa \le 2p-2$ (a special case), for the number l of solutions of the homogeneous Riemann problem there is the exact upper estimate*

$$l \le [\varkappa/2] + 1.$$

This theorem is a refinement of the known result: $l \le \varkappa + 1$ ([1–3]). In the case of the Hilbert problem for a multiply connected domain, Theorem 1 was obtained and proved by B. V. Boyarskii (([4]), addendum to Chap. IV). In the proof of Theorem 1 in the formulation given here, the ideas of B. V. Boyarskii are used.

3. Consider the Haseman problem (1) on R. In order to reduce problem (1) to the Riemann problem, we introduce a new surface R', consisting of the domains D^+ and D^- of the surface R and of a curve L', obtained by identifying boundary points $\alpha_+(Q) \in \overline{D}^+$ with boundary points $Q \in \overline{D}^-$. A point Q' of the curve L', obtained by identifying the points $\alpha_+(Q) \in \overline{D}^+$ and $Q \in \overline{D}^-$, will be denoted by $Q' \equiv \{\alpha_+(Q), Q\}$. The surface R' is homeomorphic to R and, consequently, has genus p. It is convenient to regard the surface R' as being placed over R. The symbols P', D'^+, D'^- mean that the point $P \in R-L$ and the domains D^+ and D^- are considered as objects of the surface R'. We must turn R' into a Riemann surface in such a way that from the piecewise analyticity of $\varphi^\pm(P)$ on R there would follow the piecewise analyticity of $\varphi^\pm(P')$ on R'. In the domains $D'^\pm$ we preserve the conformal structure of the surface R, reducing the parametric neighborhoods so that they do not intersect the curve L. This ensures the piecewise analyticity of the functions $\varphi^\pm(P')$. We construct at the points of the curve L' a conformal structure such that it completes the conformal structure of the domains $D'^\pm$ to a conformal structure on the whole surface R'. Let

$$Q'_0 \equiv \{\alpha_+(Q_0), Q_0\}$$

be an arbitrary point of the curve L'. Let $\zeta = f_{\alpha_+(Q_0)}(P)$ and $w = f_{Q_0}(P)$ be parametric mappings of the parametric neighborhoods $V_{\alpha_+(Q_0)}$ and V_{Q_0} of the points $\alpha_+(Q_0)$ and Q_0, respectively, in the conformal structure of the surface R. Without loss of generality, one may assume that the parametric disks of the points $\alpha_+(Q_0)$ and Q_0 coincide with the disks $K_1 : |\zeta+2| < 1$ and $K_2 : |\zeta-2| < 1$, respectively, and that

$$f_{\alpha_+(Q_0)}[\alpha_+(Q_0)] = -2; \qquad f_{Q_0}(Q_0) = 2.$$

Let l_1 be that component of the image $f_{\alpha_+(Q_0)}(L \cap V_{\alpha_+(Q_0)})$ which passes through the point $\zeta = -2$; similarly, denote by l_2 the component of the image $f_{Q_0}(L \cap V_{Q_0})$ which passes through the point $\zeta = 2$. Let $K_1^\pm$ be that component of the image $f_{\alpha_+(Q_0)}(D^\pm \cap V_{\alpha_+(Q_0)})$ which borders on l_1; similarly, denote by $K_2^\pm$ the component of the image $f_{Q_0}(D^\pm \cap V_{Q_0})$ bordering on l_2. Between certain parts of the curves l_1 and l_2 the homeomorphism $Q_1 = \alpha_+(Q)$ induces a homeomorphism $\tau = \lambda(t)$, preserving orientation in the sense that the points t and $\lambda(t)$, under motion, leave the domains K_j^+ on the same side. We now find two functions: $\Phi^+(\zeta)$, analytic and univalent in K_1^+, and $\Phi^-(\zeta)$, analytic and univalent in K_2^-, and such that

$$\Phi^+[\lambda(t)] = \Phi^-(t), \qquad t \in l_2, \quad \lambda(t) \in l_1. \tag{7}$$

This problem is easily reduced to the Haseman problem for a simply connected domain in the ζ-plane. Indeed, connect the end of the curve l_1 with the beginning of the curve l_2 and conversely by nonintersecting smooth curves in such a way that one smooth closed curve Γ is obtained, with K_1^+ lying inside Γ and K_2^- outside. Extend the homeomorphism $\lambda(t)$ in a proper way to an orientation-preserving homeomorphism of Γ onto itself. Then one may pose problem (7) on Γ and require that $\Phi^\pm(\zeta)$ have at ∞ a single-

pole of the first order. As I. B. Simonenko showed ([7]), such a problem is solvable and has a single-sheeted solution $\Phi^\pm(\zeta)$. With the aid of this solution, a parametric mapping of a neighborhood of the point $Q_0' \equiv \{\alpha_+(Q_0), Q_0\}$ is constructed as follows:

$$w = \begin{cases} \Phi^+[f_{\alpha_+(Q_0)}(P)], & P \in \overline{D}_+^- \cap V_{\alpha_+(Q_0)}, \\ \Phi^-[f_{Q_0}(P)], & P \in \overline{D}^- \cap V_{Q_0}. \end{cases}$$

If this construction is carried out at every point of the curve L', then, as is easy to show by using the single-valuedness of the functions $\Phi^\pm(\zeta)$ and the theorem on analytic continuation, the adjacency relations between the local parameters will be conformal. Thus the surface R' with the indicated conformal structure is a Riemann surface.

The boundary condition (1) on R' is transformed to the form

$$\varphi^+(Q') = G(Q')\varphi^-(Q') + g(Q'), \quad Q' \in L', \tag{8}$$

i.e., it becomes the boundary condition of the Riemann problem on R'.

Problem (2) can be reduced to a Riemann problem of the form (8) if the surface R' is defined as follows: $D^\pm \in R$, $Q \in \overline{D}^+$ is identified with $\alpha_-(Q) \in \overline{D}^-$. We transfer the conformal structure of the domain D^+ to D'^+ without change, and as local parameters of the domain D'^- we take parameters that are complex conjugate to the local parameters of the corresponding points of the domain D^-. The conformal structure on L' is then constructed in the same way as in problem (1). Since in the planar Haseman problem it suffices to have smoothness of the contour for the known qualitative results to hold, problem (2) is also reduced to a problem of the form (8) under the condition that L be smooth. This remark also applies to problems (3)–(6). Using the results of papers ([1]-[3]), Theorem 1, and the indicated possibility of reducing problems (1) and (2) to (8), we obtain, for example, the following theorem.

Theorem 2. Let $\varkappa = \operatorname{ind} G(Q)$, $g(Q) \equiv 0$, and let l be the number of solutions of problem (1) or (2). We have, for $\varkappa < 0$, $l = 0$; for $\varkappa > 2p - 2$, $l = \varkappa - p + 1$. For $0 \le \varkappa < 2p - 2$ we have the sharp estimate

$$\max\{0, \varkappa - p + 1\} \leq l \leq [\varkappa/2] + 1.$$

When reducing problems (3) and (4) to the Riemann problem, the surface R' is constructed as follows. Let D_1 and D_2 be two copies of the domain D^+. Assuming that $D_1 \in R'$, $D_2 \in R'$, we glue the point $\alpha_+(Q) \in \overline{D}_1$ to the point $Q \in \overline{D}_2$. Just as in problems (1) and (2), here one can construct the required conformal structure, after which problems (3) and (4) are reduced to the Riemann problem on a surface R of genus $p = m + 2h$, if the function $\varphi(P)$ is sought on D_1, and the function $\psi(P)$ on D_2. For problem (3), Theorem 2 is valid for $p = m + 2h$; for problem (4), Theorem 2 is valid if, in addition, $\varkappa$ is replaced by $(-\varkappa)$.

Thus problems (1)–(4) can be reduced to Riemann problems equivalent to them on Riemann surfaces. Consequently, the theory of the problems with shift under consideration is included in the theory of the Riemann problem on abstract Riemann surfaces.

It is easy to write down explicitly the boundary conditions of the homogeneous problems for the covariants associated with problems (1)–(4), in terms of the solutions of which the solvability conditions for the nonhomogeneous problems (1)–(4) are written.

4. We shall consider problem (5) as problem (3) with $\varphi \equiv \psi$. Assuming the conditions

$$\overline{G(Q)}\, G[\alpha_+(Q)] \equiv 1, \qquad \overline{g(Q)}\, G[\alpha_+(Q)] + g[\alpha_+(Q)] \equiv 0,$$

to be satisfied, one can obtain the following theorem.

Theorem 3. The fundamental system of solutions $\{\varphi_1, \psi_1\}, \ldots, \{\varphi_l, \psi_l\}$ of the homogeneous problem (3) can always be chosen so that $\varphi_k = \psi_k$, $k =$

$= 1, \ldots, l$. If $C_1, \ldots, C_l$ are arbitrary constants, then the pair of linear combinations

$$\varphi = \sum_{k=1}^{l} c_k \varphi_k, \qquad \psi = \sum_{k=1}^{l} \bar{c}_k \varphi_k$$

is the general solution of problem (3), if c_k are complex numbers, and the general solution of problem (5), if c_k are real numbers.

5. We consider problem (6) as problem (4) with $\varphi \equiv \psi$. Let $g(Q) \equiv 0$ in (4) and (6), $G(Q)\, G[\alpha_-(Q)] \equiv 1$.

Theorem 4. The equality $l = l_1 + l_2$ is valid, where l_1 is the number of solutions of problem (6), and l_2 is the number of solutions of the problem

$$\varphi[\alpha_-(Q)] = -G(Q)\varphi(Q).$$

From the reducibility of problem (3) to the Riemann problem it follows that solutions of the homogeneous problems (1)–(6) can have in $\overline{D}^{\pm}$ only a finite number of zeros, and all zeros have integer finite multiplicity. This makes it possible to apply the principle of the argument to all the problems under consideration. Then, using the peculiarities in the arrangement of the zeros, it is sometimes possible to count the number of solutions. For example, suppose that in problem (6) $\alpha\_(Q)$ maps each curve of the contour L onto itself and suppose $h = 0$. Then the following theorem is valid.

Theorem 5. For $-\varkappa > 2m - 2$, the number of solutions of the homogeneous problem (5) is

$$l = 1 - \frac{\varkappa + m^2}{2},$$

where m is the number of fixed points of the transformation (α_Q) at which $G(Q) = -1$.

This result generalizes the well-known result of D. A. Kveselava [6].

Rostov-on-Don
State University

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Note: Figure translations are in progress. See original paper for figures.