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Abstract

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ON THE THEORY OF MULTISTAGE ROCKETS

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The problem of the optimal selection of the stages of a multistage rocket is considered. From the mathematical point of view, the problem reduces to a variational problem of the Bolza–Mayer type. A distinctive feature of problems of this kind is that the form of the control functions on the intervals of variation between the points of discontinuity is known. The positions of the discontinuity points of the control functions are found from the extremum conditions for a certain functional. A concrete calculation is carried out for the example of the motion of a two-stage rocket in a homogeneous gravitational field without taking account of the resistance of the medium.

1°. Let the behavior of a certain dynamical system (a multistage rocket) be described by ordinary differential equations of first order:

$$g_s = \dot{x}_s - f_s(x_1, \dots, x_n, u_1, \dots, u_m, t) = 0 \qquad (s = 1, \dots, n) \tag{1,1}$$

and by a system of finite relations

$$\psi_k = \psi_k(u_1, \dots, u_m, t) = 0 \qquad (k = 1, \dots, r < m). \tag{1,2}$$

In equations (1,1), (1,2), $x_s(t)$ are the coordinates determining the position of the dynamical system, and $u_j(t)$ are the control functions, having discontinuities of the first kind at the instants t_i, where $m-r$ control functions $u_1, \dots, u_{m-r}$ are given in the form of explicit dependences on the time t and on the parameters t_i. At the initial instant of time $t = t_0$, the position of the system is determined by the values of the coordinates $x_s(t_0) = x_s^0$ $(s = 1, \dots, n)$ and by the values of the control functions $u_j(t_0) = u_j^0$ $(j = 1, \dots, m)$.

The coordinates of the system at the instant $t = T$ are connected by the equalities

$$\Phi_l = \Phi_l[x_s(T), T] = 0 \qquad (l = 1, \dots, p < n). \tag{1,3}$$

It is required to determine such instants of time t_i at which the functional $J = J[x_s(T), T]$ attains an extremum, under satisfaction of the equalities (1,1) –(1,2).

2°. We form the auxiliary functional

$$I = J + \sum_{l=1}^{p} \rho_l \Phi_l + \int_{t_0}^{T} \left\{ \sum_{s=1}^{n} \lambda_s g_s + \sum_{k=1}^{r} \mu_k \psi_k \right\} dt. \tag{2,1}$$

In expression (2,1), $\lambda_s(t)$, $\mu_k(t)$, ρ_l are undetermined Lagrange multipliers.

Suppose that in the time interval under consideration (t_0, T) there exist two points t_1, t_2 at which the control functions suffer discontinuities, $t_0 < t_1 < t_2 < T$. We shall denote the values of the functions introduced above and considered in the interval (t_0, t_1) by the index 1 (for example, $x_s^{(1)}(t)$, $u_j^{(1)}(t)$); in the interval (t_1, t_2) by the index 2 (for example, $x_s^{(2)}(t)$, $u_j^{(2)}(t)$); in the interval (t_2, T) by the index 3.

In computing the variation of the functional I, we shall take the following circumstances into account. The presence of discontinuities in the control functions compels

makes it possible to consider the change of the points of discontinuity t_i. If, for example, the terminal time is not fixed (T is free), then between the "variation of the end point" and the "variation at the end" there is the dependence

$$\Delta x_s^{(3)}(T) = \delta x_s^{(3)}(T) + \dot{x}_s^{(3)}(T)\,\delta T. \tag{2,2}$$

For variations of the expressions J, $\sum_{l=1}^{p} \rho_l \Phi_l$, analogous equalities are obtained.

The initial data determining the position of the dynamical system are known; the trajectory $x_s^{(1)}(t)$ can be found from the solution of the corresponding system of equations and, consequently, $\delta x_s^{(1)}(t_1) = 0$, i.e. an equality similar to (2,2) has the form

$$\Delta x_s^{(1)}(t_1) = \dot{x}_s^{(1)}(t_1)\,\delta t_1.$$

The "variation at the end" $\delta x_s^{(1)}(t_1)$ and the "variation of the end point" $\Delta x_s^{(2)}(t_1)$ of the intermediate trajectory are related by the relation

$$\Delta x_s^{(2)}(t_1) = \delta x_s^{(2)}(t_1) + \dot{x}_s^{(2)}(t_1)\,\delta t_1.$$

Fig. 1

Fig. 1. Diagram of stages: labels include $u = m/m_0$, u_1-, u_1, u_2-, u_2, u_n-, u_n, t_1, t_2, T, t; "I stage," "II stage," "n-th stage," "payload" ; and vertical labels "I subrocket," "II subrocket," "n-th subrocket."

Figure 1: Fig. 1. Diagram of stages: labels include $u = m/m_0$, u_1-, u_1, u_2-, u_2, u_n-, u_n, t_1, t_2, T, t; "I stage," "II stage," "n-th stage," "payload" ; and vertical labels "I subrocket," "II subrocket," "n-th subrocket."

Owing to the continuity of the trajectory $x_s(t)$ at the points of discontinuity of the control $u_j(t)$, we have

$$\Delta x_s^{(1)}(t_1) = \Delta x_s^{(2)}(t_1).$$

Consequently,

$$\delta x_s^{(2)}(t_1) = \left[\dot{x}_s^{(1)}(t_1) - \dot{x}_s^{(2)}(t_1)\right] \delta t_1.$$

For variations of the right-hand curve $x_s^{(3)}(t)$ at the point t_2, we find

$$\delta x_s^{(3)}(t_2) = \delta x_s^{(2)}(t_2) + \left[\dot{x}_s^{(2)}(t_2) - \dot{x}_s^{(3)}(t_2)\right] \delta t_2.$$

The form of the dependences $u_j(T, t, t_i)$, determined in each specific problem by its conditions, does not affect the course of the proof; for definiteness let us put $u_j^{(2)}(t, t_1)$, $u_j^{(3)}(t, t_1, t_2)$. Then the variations of the control functions will have the form

$$\delta u_j^{(2)} = \frac{\partial u_j^{(2)}}{\partial t_1} \delta t_1; \qquad \delta u_j^{(3)} = \frac{\partial u_j^{(3)}}{\partial t_1} \delta t_1 + \frac{\partial u_j^{(3)}}{\partial t_2} \delta t_2.$$

Forming the variation of the functional I, taking into account the remarks made, we note that the variations $\delta x_s^{(2)}(t)$, $\delta x_s^{(3)}(t)$, $\delta x_s^{(2)}(t_2)$, $\delta \lambda_s^{(2)}(t)$, $\delta \lambda_s^{(3)}(t)$, $\delta \mu_k^{(2)}(t)$, $\delta \mu_k^{(3)}(t)$ $(k = 1, \dots, r)$, δt_1, δt_2, δT, and $n-p$ variations $\delta x_s^{(3)}(T)$ are independent.

Defining $2r$ multipliers $\mu_k^{(2)}(t)$, $\mu_k^{(3)}(t)$ so that the coefficients of the variations $\delta u_\nu^{(2)}$, $\delta u_\nu^{(3)}$ $(\nu = m - r + 1, \dots, m)$ vanish, and p multipliers ρ_l so that the coefficients of the dependent variations $\delta x_s^{(3)}(T)$ vanish, we set the coefficients of the remaining independent variations equal to zero. As a result we obtain the equations satisfied by the coordinates of the system and the control functions:

$$\frac{dx_s^i}{dt} = f_s(x^i, u^i, t); \quad \psi_k^i(u^i, t) = 0 \quad (s = 1, \dots, n;\ k = 1, \dots, r;\ i = 2, 3); \quad (2,3)$$

the differential equations satisfied by the functions $\lambda_s^i(t)$:

$$\dot{\lambda}_s^i + \sum_{\alpha=1}^{n} \lambda_\alpha^i \frac{\partial f_\alpha}{\partial x_s^i} = 0 \quad (s = 1, \dots, n;\ i = 2, 3). \qquad (2{,}4)$$

Boundary conditions for the functions $\lambda_s^{(3)}(T)$:

$$\lambda_s^{(3)}(T) + \frac{\partial}{\partial x_s^{(3)}(T)} \left[J + \sum_{l=1}^{p} \rho_l \Phi_l \right] = 0 \quad (s = 1, \dots, n). \qquad (2{,}5)$$

A boundary condition of the form

$$\frac{d}{dT} \left[J + \sum_{l=1}^{p} \rho_l \Phi_l \right] = 0. \qquad (2{,}6)$$

Continuity conditions for the functions $\lambda_s(t)$

$$\lambda_s^{(2)}(t_2) = \lambda_s^{(3)}(t_2) \quad (s = 1, \dots, n). \qquad (2{,}7)$$

Equations for determining the multipliers $\mu_k^{(2)}(t)$, $\mu_k^{(3)}(t)$:

$$\sum_{\alpha=1}^{n} \lambda_\alpha^i \frac{\partial f_\alpha}{\partial x_s^i} - \sum_{k=1}^{r} \mu_k^i \frac{\partial \psi}{\partial u_k^i} = 0 \quad (k = 1, \dots, r;\ i = 2, 3). \qquad (2{,}8)$$

Conditions at the discontinuity points of the controls t_i

$$\begin{gathered} \sum_{s=1}^{n} \lambda_s^{i+1}(t_i) \left[\dot{x}_s^i(t_i) - \dot{x}_s^i(t_i) \right] + \\ + \int_{t_i}^{T} \left\{ \sum_{k=1}^{m-r} \left[\sum_{s=1}^{n} \lambda_s \frac{\partial f_s}{\partial u_k} - \sum_{\beta=1}^{r} \mu_\beta \frac{\partial \psi_\beta}{\partial u_k} \right] \frac{\partial u_k}{\partial t_i} \right\} dt = 0 \quad (i = 1, 2). \qquad (2{,}9) \end{gathered}$$

The equations found, (2,3)–(2,9), as is easy to see, solve the problem posed.

3°. As an example, consider the calculation of a two-stage rocket moving in a uniform gravitational field and in space where the action of aerodynamic forces is not taken into account. We shall find such a time t_1 for the transition from the first stage of the composite rocket to the second, at which a certain functional would attain its maximum for a fixed flight time of the rocket on the powered segment.

The following formulations of the problem are possible:

a) $J = x(T)$, which corresponds to the maximum range of the powered segment; b) $J = h(T)$, which corresponds to the maximum altitude; c) $J = v(T)$, which corresponds to the maximum velocity at the end of the powered segment; d) $J = v(T)\cos\theta(T)$, which corresponds to the maximum horizontal component of the velocity at the end of the powered segment.

The equations of plane-parallel motion of the center of mass of the composite rocket have the form

$$g_1 = \dot{v} + g\sin\theta + V^r\frac{\dot{m}}{m} = 0; \qquad g_2 = \dot{\theta} - \frac{\cos\theta}{v} = 0, \tag{3,1}$$

$$g_3 = \dot{x} - v\cos\theta = 0; \qquad g_4 = \dot{h} - v\sin\theta = 0,$$

where g is the acceleration of gravity, a constant quantity; v is the velocity of the center of mass of the composite rocket; m is the mass of the rocket, changing according to a linear law; θ is the angle of inclination of the velocity vector to the horizon; V^r is the relative velocity of the expelled particles; x is the range; h is the altitude of the rocket' s flight.

Introduce the dimensionless mass of the staged rocket $u = m/m_0$, where m_0 is the initial weight of the staged rocket.

If one considers the auxiliary plane $\{u, t\}$, then the segments of the curve $u = u(t)$ corresponding to the operating regime of the engines are represented by inclined straight lines, and the segments of stage separation by vertical cuts (Fig. 1).

Let us denote the ratio of the "dry-weight mass" of the i-th $(1, 2)$ stage to the mass of its fuel by k_i. In addition, let $u_{i-} = u(t_i{-}0)$, $u_i = u(t_i{+}0)$. Analytically, the function u is determined by the $2n$ equalities

$$u_{i-} = u_{i-1} - \beta_i(t_i - t_{i-1}); \quad u_i = u_{i-}(1 + k_i) - k_i u_{i-1} \quad (i = 1, 2), \tag{3,2}$$

where β_i is the fuel consumption per second of the engine of the i-th stage. Note that $t_0 = 0,\ u_0 = 1,\ t_n = T,\ u_n = m_p/m_0;\ m_p$ is the payload mass.

The engines of successive stages operate without pauses.

For a two-stage rocket we have the following equalities:

$$\frac{\partial u_{1-}}{\partial t_1} = -\beta_1; \quad \frac{\partial u_{2-}}{\partial t_1} = -\beta_2 k_2; \tag{3,3}$$

$$\frac{\partial u_1}{\partial t_1} = -\beta_1(1 + k_1); \quad \frac{\partial u_2}{\partial t_1} = \beta_2(1 + k_2) - \beta_1(1 + k_1) = 0.$$

The system of differential equations (2,6) for determining $\lambda_s(t)$ (here and below $\lambda_s^{(2)}(t) \equiv \lambda_s(t)$; $v_s^{(2)}(t) \equiv v_s(t)$, etc.) will have the form

$$\dot{\lambda}_1 = \lambda_2 \left[-\frac{g}{v^2} \cos\theta \right] - \lambda_3 \cos\theta - \lambda_4 \sin\theta, \tag{3,4}$$

$$\dot{\lambda}_2 = \lambda_1 g \cos\theta - \lambda_2 \frac{g}{v} \sin\theta + \lambda_3 v \sin\theta - \lambda_4 v \cos\theta; \quad \dot{\lambda}_3 = 0; \quad \dot{\lambda}_4 = 0.$$

The system of differential equations (3,4) admits the first integrals (3):

$$\frac{\lambda_2}{v} \cos\theta + \lambda_1 \sin\theta = -\lambda_4 t + C_1; \quad \lambda_1 \cos\theta - \frac{\lambda_2}{v} \sin\theta = -\lambda_3 t + C_2; \tag{3,5}$$

$$\lambda_3 = C_3, \quad \lambda_4 = C_4.$$

Condition (2,9), taking account of the equalities (3,3), gives

$$\lambda_1(t_1)\left[\dot{v}(t_1 - 0) - \dot{v}(t_1 + 0)\right] + \beta_2^2 V_2^r k_2 \int_{t_1}^{T} \frac{\lambda_1(t)}{u^2}\, dt = 0. \tag{3,6}$$

From equations (3,5) we find

$$\lambda_1(t) = [-\lambda_3 t + C_2] \cos\theta + [-\lambda_4 t + C_1] \sin\theta.$$

If, for example, case c) is considered, then from equations (2,5) we obtain

$$\lambda_1(T) = -1, \quad \lambda_2(T) = \lambda_3(T) = \lambda_4(T) = 0,$$

and hence

$$\lambda_3 = \lambda_4 = 0, \quad \lambda_1(t) = -\cos[\theta - \theta(T)].$$

If case d) is considered, then equations (2,5) give

$$\lambda_1(T) = -\cos\theta(T); \quad \lambda_2(T) = v(T) \sin\theta(T); \quad \lambda_3(T) = \lambda_4(T) = 0.$$

Taking the equalities (3,5) into account, we find:

$$C_1 = C_3 = C_4 = 0; \quad C_2 = -1; \quad \lambda_1(t) = -\cos\theta.$$

If the vertical flight of the rocket is considered, then condition (3,6), taking account of the equalities (3,2), gives

$$V_1^{r0} \beta_1 u_1 u_{2-} = V_2^{r0} \beta_2 u_{1-} u_2. \tag{3,7}$$

Equations (3,6), (3,7) determine the required instant of time t_1.

Thus, the calculation of any multistage rocket can be carried out.

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Note: Figure translations are in progress. See original paper for figures.

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