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**Abstract**

**Full Text**

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## ASYMPTOTIC DISTRIBUTION OF THE EIGENVALUES OF ELLIPTIC OPERATORS

*(Presented by Academician I. G. Petrovskii on 4 IV 1964)*

It is known that the Schrödinger operator  $-\Delta + q(x)$ , defined in the whole space  $R_3$ , has a discrete spectrum if  $q(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ . Titchmarsh <sup>(1)</sup>, B. M. Levitan <sup>(2)</sup>, and Ray <sup>(3)</sup> established that, under certain additional conditions on  $q(x)$ , the number of eigenvalues  $N(\lambda)$  less than  $\lambda$  has the following simple asymptotics:

$$N(\lambda) \sim \frac{1}{6\pi^2} \int_{q(x) < \lambda} (\lambda - q)^{3/2} dx.$$

In the present work similar formulas are established for arbitrary self-adjoint elliptic operators.

1. Consider the elliptic operator

$$\begin{aligned} L &= (-1)^m \sum_{k_1 + \dots + k_n = 2m} A^{k_1 \dots k_n}(x) \frac{\partial^{2m}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} + L_1 \left( x, \frac{\partial}{\partial x} \right) + q(x) = \\ &= L_0 + L_1 + q(x). \end{aligned}$$

Here  $x = (x_1, \dots, x_n)$ ,  $-\infty < x_i < \infty$ ,  $A^{k_1 \dots k_n}(x)$  are functions bounded in the whole space  $R_n$  and satisfying a Lipschitz condition:

$$|A^{k_1 \dots k_n}(x) - A^{k_1 \dots k_n}(\xi)| \leq K|x - \xi|^\gamma, \quad \text{if } |x - \xi| < 1,$$

$\gamma > 0$ , and the constant  $K$  does not depend on  $x$  and  $\xi$ . In addition, it is assumed that the form of the highest terms is uniformly elliptic.

The operator  $L_1 \left( x, \frac{\partial}{\partial x} \right)$  contains derivatives of order  $< 2m$ , and its coefficients are bounded in the whole space. The function  $q(x) \geq 1$  and satisfies the following conditions:

- 1)  $|q(x) - q(\xi)| \leq Bq^\alpha(\xi)|x - \xi|$  for  $|x - \xi| \leq 1$ ,  $\alpha < 1 + \frac{1}{2m}$ .
- 2)  $q(x) \leq B \exp [C_0|x - \xi|q^{1/2m}(\xi)]$  for  $|x - \xi| > 1$ .
- 3)  $\int_{|x-\xi| \leq 1} \exp[-aq(\xi)] d\xi \leq B \exp \left[ -\frac{a}{k_0}q(x) \right]$  for some  $k_0 \geq 1$ .
- 4)  $q(x) \leq Aq(\xi)$  for  $|x - \xi| \leq 1$ .
- 5)  $\int_{R_n} \frac{dx}{q^\beta(x)} < \infty$  for some  $\beta > 0$ .

Define the function

$$M(\lambda) = \int_{q(x) < \lambda} \Phi(x)(\lambda - q)^{n/2m} dx, \quad \text{where } \Phi(x) = \\ = \int_{-\infty}^{\infty} e^{-L_0(x,s)} ds,$$

and  $L_0(x, s)$  is the homogeneous form corresponding to the group of highest terms

$$L_0 \left( x, \frac{\partial}{\partial x} \right).$$

We shall assume that, for sufficiently large  $\lambda_1, \lambda_2$ , the condition

$$6) \quad \frac{M(\lambda_1)}{M(\lambda_2)} \leq C \left( \frac{\lambda_1}{\lambda_2} \right)^{\gamma_1}, \quad \gamma_1 > 0, \quad \lambda_1 > \lambda_2.$$

**Theorem 1.** If the elliptic operator  $L \left( x, \frac{\partial}{\partial x} \right)$  satisfies the requirements formulated above, then its spectrum is discrete and the number of eigenvalues  $N(\lambda)$  less than  $\lambda$  has an asymptotic expression of the form

$$N(\lambda) \sim \frac{1}{(2\pi)^n \Gamma(n/2m + 1)} \int_{q(x) < \lambda} \Phi(x)(\lambda - q(x))^{n/2m} dx. \quad (1)$$

Let us note that if the coefficients of the highest derivatives do not depend on  $x$ , then  $\Phi(x)$  is a constant.

For the equation of the second order,

$$\Phi(x) = (\det \|a_{ij}(x)\|)^{-n/2}.$$

**Remark 1.** One may also write the asymptotics of the weighted trace  $N_s(\lambda)$ :

$$N_s(\lambda) = \sum_{\lambda_n < \lambda} \int_{R_n} q^s(x) \varphi_n^2(x) dx,$$

where  $\varphi_n(x)$  are the eigenfunctions of the operator  $L$ . This asymptotic formula is valid under the previous conditions and has the form

$$N_s(\lambda) \sim \frac{1}{(2\pi)^n \Gamma\left(\frac{n}{2m} + 1\right)} \int_{q(x) < \lambda} \Phi(x) q^s(x) (\lambda - q(x))^{n/2m} dx.$$

**Remark 2.** Theorem 1 is also valid for so-called quasi-elliptic operators, which are somewhat more general than elliptic ones.

For example, the function  $N(\lambda)$  of the operator

$$\sum_{i=1}^n (-1)^{m_i} \frac{\partial^{2m_i}}{\partial x_i^{2m_i}} + q(x)$$

under the previous restrictions on  $q(x)$  has principal term

$$N(\lambda) \sim \frac{M}{(2\pi)^n \Gamma\left(\sum \frac{1}{2m_i} + 1\right)} \int_{q(x) < \lambda} (\lambda - q(x))^{\frac{1}{2} \sum \frac{1}{m_i}} dx,$$

where the constant  $M$  is equal to

$$\int_{-\infty}^{\infty} \exp\left[-\sum s_i^{2m_i}\right] ds.$$

Let us note that, as is not difficult to verify, conditions 2)–5) are satisfied if  $q(x)$  is enclosed between two paraboloids:

$$B_1(1 + |x|^k) \leq q(x) \leq B_2(1 + |x|^k), \quad k > 0.$$

Condition 1) will be satisfied, for example, if  $q(x)$  has a derivative  $q'(x)$  satisfying the estimate

$$|q'(x)| \leq Cq^\alpha(x), \quad \alpha < 1 + \frac{1}{2m}.$$

**2.** Let us briefly outline the proof of the theorem. The Green function  $G(x, y, t)$  of the Cauchy problem for the parabolic equation

$$\frac{\partial u}{\partial t} = -Lu \quad (2)$$

is connected with the spectral function  $\theta(x, y, \lambda)$  of the elliptic operator  $L \geq 0$  by the simple relation

$$G(x, y, t) = \int_0^\infty e^{-\lambda t} d\theta(x, y, \lambda).$$

If the spectrum is discrete, then from this equality it immediately follows that

$$\int_{-\infty}^\infty G(x, x, t) dx = \int_0^\infty e^{-\lambda t} dN(\lambda).$$

Thus the problem reduces to obtaining the asymptotics of the Green's function  $G(x, y, t)$  for small  $t$ , uniformly in  $x$  in the whole space  $R_n$ . It should be said that writing down the asymptotics for small  $t$  in each bounded domain  $S \subset R_n$  is not difficult. It depends only on the group of highest derivatives and does not take into account the decisive influence of  $q(x)$ . We shall seek the Green's function  $G(x, y, t)$  by the "parametrix" method in the form

$$G(x, y, t) = G_0(x - y, y, t)e^{-tq(y)} + \int_0^t d\tau \int_{-\infty}^\infty G_0(x - \xi, \xi, t - \tau)e^{-(t-\tau)q(\xi)}\varphi(\xi, y, \tau) d\xi.$$

Here  $G_0(x - y, \eta, t)$  is the Green's function of the parabolic equation

$$\frac{\partial u}{\partial t} = -L_0 \left( \eta, \frac{\partial}{\partial x} \right) u.$$

From the fact that  $G(x, y, t)$  is the Green's function of equation (2), we obtain that the kernel  $\varphi(x, y, t)$  satisfies the integral equation

$$\varphi(x, y, t) - K_0(x, y, t) = \int_0^t d\tau \int_{-\infty}^\infty K_0(x, \xi, t - \tau)\varphi(\xi, y, \tau) d\xi, \quad (3)$$

where

$$K_0(x, y, t) = \left\{ L \left( x, \frac{\partial}{\partial x} \right) - L \left( y, \frac{\partial}{\partial x} \right) \right\} G_0(x - y, y, t)e^{-tq(y)}.$$

It is not hard to verify that, when the conditions of item 1 are fulfilled, the kernel  $K_0(x, y, t)$  satisfies the inequalities

$$|K_0(x, y, t)| \leq \frac{A \exp \left[ -c \frac{|x-y|^r}{t^{1/(2m-1)}} \right]}{t^{(n+2m-\varepsilon)/2m}} \exp[-c_1 tq(y)] \quad \text{for } |x-y| \leq 1,$$

$$|K_0(x, y, t)| \leq \frac{A \exp \left[ -c \frac{|x-y|^r}{t^{1/(2m-1)}} \right]}{q^l(y)} \exp[-c_1 tq(y)] \quad \text{for } |x-y| > 1.$$

Here  $A, c, c_1$  are some constants  $> 0$ ;  $l$  is an arbitrarily large integer;  $r$  is the number conjugate to  $2m$ , i.e.  $\frac{1}{r} + \frac{1}{2m} = 1$ ,  $\varepsilon > 0$ .

It turns out that the integral equation (3) can be solved by the method of successive approximations. Moreover, up to a certain number  $k$ , the iterated kernels

$$K_j(x, y, t) = \int_0^t d\tau \int_{-\infty}^{\infty} K_0(x, \xi, t-\tau) K_{j-1}(\xi, y, \tau) d\xi$$

will satisfy the estimate

$$|K_j(x, y, t)| \leq \begin{cases} \frac{A_j \exp \left[ -c_j \frac{|x-y|^r}{t^{1/(2m-1)}} \right]}{t^{n/2m-j\delta}} \exp[-c_{jt}q(y)] & \text{for } |x-y| \leq 1, \\ \frac{A_j \exp \left[ -c_j \frac{|x-y|^r}{t^{1/(2m-1)}} \right]}{q^l(y)} \exp[-c_{jt}q(y)] & \text{for } |x-y| > 1; \end{cases} \quad (4)$$

where  $c_j > 0$ ,  $\delta > 0$ .

It follows from inequality (4) that, for sufficiently large  $j$ ,  $j > n/2m\delta = j_0$ , the kernel  $K_j(x, y, t)$  will have no singularity with respect to  $t$ . Moreover, for  $j$  greater than some  $j'_0$ , the inequality

$$|K_j(x, y, t)| \leq A_j \frac{\exp \left[ -c_j \frac{|x-y|^r}{t^{1/(2m-1)}} \right]}{q^l(y)} \quad \text{for all } x \text{ and } y \quad (5)$$

will hold.

Starting from this point, we shall estimate the iterated kernels differently: we assume that

$$|K_0(x, y, t)| \leq \frac{A \exp \left[ -c_0 \frac{|x-y|^r}{t^{1/(2m-1)}} \right]}{t^{(n+2m-\varepsilon)/2m}}.$$

We shall estimate the kernels  $K_j(x, y, t)$  according to inequality (5). The iterated kernels, however, will be estimated as S. D. Eidelman did (4). As a result, we obtain that the Green's function has the form

$$G(x, y, t) = G_0(x - y, t)e^{-tq(y)} + \frac{e^{-ctq(y)}}{t^{n/2m-\varepsilon_0}}O(1) + \frac{O(1)}{t^{n/2m-\varepsilon_0}q^l(y)}. \quad (6)$$

From this inequality we obtain

$$g(t) = \int_{-\infty}^{\infty} G(x, x, t) dx = \frac{1}{t^{n/2m}} \int_{-\infty}^{\infty} \Phi(x)e^{-tq(x)} dx + \frac{O(1) \int_{-\infty}^{\infty} e^{-ctq(x)} dx}{t^{n/2m-\varepsilon}}.$$

Next, note that

$$g_0(t) = \frac{1}{t^{n/2m}} \int_{-\infty}^{\infty} \Phi(x)e^{-tq(x)} dx = \frac{1}{\Gamma(n/2m + 1)} \int_0^{\infty} e^{-\lambda t} dM(\lambda).$$

If one assumes that

$$\int_{-\infty}^{\infty} e^{-tq(x)} dx \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-ctq(x)} dx$$

have singularities at zero in  $t$  of the same order, then  $g(t) \sim g_0(t)$ . Applying now the Tauberian theorem of B. I. Korenblum (5), we obtain formula (1). We note that condition 6) of Sec. 1 is the "Tauberian condition" necessary for the application of the Tauberian theorem.

As for the discreteness of the spectrum of the operator  $L$ , it can likewise be obtained from the representation of the Green's function  $G(x, y, t)$  in the form (6), since the kernel  $H(x, y, \lambda)$  of the resolvent  $R_\lambda = (L - \lambda E)^{-1}$  is related to the function  $G(x, y, t)$  by the formula

$$H(x, y, \lambda) = \int_0^{\infty} G(x, y, t)e^{-\lambda t} dt.$$

It will follow from estimate (6) that  $H(x, y, \lambda)$  defines an operator some power of which is a Hilbert-Schmidt operator. From the estimate obtained in (6) it also follows that the eigenfunctions decrease faster than  $c_k/q^k(x)$  for any  $k > 0$ . The assertions stated in Remarks 1 and 2 are proved according to the scheme indicated above. It would be possible to dispense with the requirement that the coefficients of the operator  $L_1$  be bounded and to allow a certain growth at infinity, coordinated in a definite way with the growth of  $q(x)$ . Formula (1) remains unchanged in that case.

In conclusion we note that conditions 1)–5) of Sec. 1 are satisfied mainly by functions of power growth. If, however,  $q(x)$  grows at infinity faster than any power, then formula (1) is also preserved; moreover, the required asymptotics for the Green's function  $G(x, y, t)$  can be obtained still more simply <sup>(6)</sup>, if one requires the existence for the function  $q(x)$  of derivatives of order  $2m$  satisfying the inequalities:

$$|q^{(i)}(x)| \leq c_i q^{(2m+i)/2m}(x), \quad i = 1, 2, \dots, 2m.$$

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*Note: Figure translations are in progress. See original paper for figures.*

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