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## Abstract

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# MATHEMATICS

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# ON THE EXISTENCE OF ABELIAN SUBGROUPS OF INFINITE RANK IN LOCALLY SOLVABLE GROUPS

*(Presented by Academician A. I. Mal' cev, 22 I 1964)*

A. I. Mal' cev ($^1$), V. S. Charin ($^{2,3}$), and M. I. Kargapolov ($^4$) have devoted their works to the solution of the important question of the existence of abelian subgroups of infinite rank in locally solvable groups. In the present note the following has been obtained.

**Theorem 1.** *A periodic locally solvable group of infinite rank has an abelian subgroup of infinite rank.*

The following lemma, needed for the proof of the theorem just formulated, is of independent interest. In the proof of the lemma the following notation is used: $G_0 = GL(r, p^n)$ is the group of all matrices of degree $r$ over the residue ring modulo $p^n$, whose determinants are not zero divisors; $G_i,\ i = 1, 2, \dots, n-1$, is the set of matrices in $GL(r, p^n)$ of the form $E + p^i A$, where $E$ is the identity matrix. Obviously, $G_0/G_i \cong GL(r, p^i)$, $i = 1, 2, \dots, n-1$.

**Lemma 1.** *If $r$ is the rank of some maximal abelian normal divisor $A$ of a finite $p$-group $P$, then the rank of $P$ does not exceed $3r^2$.*

**Proof.** It is clear that the factor group $P/A$ is isomorphic to a factor group of some subgroup of $GL(r, p^n)$, where $p^n$ is the exponent of $A$. Therefore it is enough to show that the rank of an arbitrary $p$-subgroup $H$ of $GL(r, p^n)$ does not exceed $3r^2 - r$.

Since $G_0/G_1 \cong GL(r, p)$, the order of its Sylow $p$-subgroup is $p^{r(r-1)/2}$, and therefore the rank of the latter does not exceed $r(r-1)/2 \le r^2 - r$. The order of the abelian elementary group $G_1/G_2$ does not exceed $p^{r^2}$, and therefore its rank is not greater than $r^2$. Thus it remains to prove that the rank of $G_2$ is not greater than $r^2$.

Let $S$ be an arbitrary subgroup of $G_2$, and let $s$ be the rank of $S \cap G_{n-1}$. We shall prove by induction on $n$ that $S$ has a system of generators consisting of $s$ elements.

Let us note that every element of order $p$ from $G_2$ belongs to $G_{n-1}$. Indeed, if $(E + p^i X)^p = E$, then $p^{i+1} X \equiv 0 \pmod{p^{2i}}$, where 0 is the zero matrix, and therefore for $i > 1$, $X \equiv 0 \pmod{p}$. But this means that if $(E + p^2X)^p = E$, then $X \equiv 0 \pmod{p^{n-3}}$.

If $n = 3, 4$, then $G_2$ is an abelian group, and the rank of $S$, in view of the assertion of the preceding paragraph, coincides with the rank of $S \cap G_{n-1}$ and hence is equal to $s$. Let $n > 4$ and let $D(S \cap G_{n-2})$ be the Frattini subgroup of the group $S \cap G_{n-2}$. Then, since $G_{n-2}$ is an abelian group and since all its elements of order $p$ are contained in $G_{n-1}$, $S$ and $S/D(S \cap G_{n-2})$ have the same number of generators. If $s - t$ is the rank of $S \cap G_{n-2}/S \cap G_{n-1}$, then, by the induction hypothesis, $S/S \cap G_{n-1}$, as a subgroup of $GL(r, p^{n-1})$, has a system of $s - t$ generators. Since $S \cap G_{n-1}/D(S \cap G_{n-2})$ has $t$ generators, $S/D(S \cap G_{n-2})$ has a system of $s$ generators. As was noted, in this case $S$ also has $s$ generators. Since, obviously, $s \leq r^2$, $G_2$ has rank at most $r^2$. The lemma is proved.

By virtue of a result of A. I. Mal' cev (see ([1])) on the finiteness of the rank of a locally nilpotent torsion-free group all of whose abelian subgroups have finite rank, Lemma 1 implies

**Theorem 2.** *A locally nilpotent group of infinite rank has an abelian subgroup of infinite rank.*

In what follows we shall need the following.

**Lemma 2.** *A finite solvable group $G$, whose abelian subgroups have rank at most $r$, has rank not exceeding a certain number $f(r)$, depending only on $r$.*

This assertion follows immediately from Lemma 1 and Zassenhaus' theorem on the solvability class of a solvable matrix group (see ([5])).

Now let us consider the special case of locally solvable groups.

**Lemma 3.** *A periodic locally solvable group $G$ of infinite rank, all Sylow $p$-subgroups of which (for all $p$) are finite, has an abelian subgroup of infinite rank.*

**Proof.** From Lemma 2 and the infiniteness of the rank of the group $G$ it follows that the ranks of the abelian subgroups of $G$ are not bounded in the aggregate. Thus, for every natural number $r$ there exists a prime number $p_r$ such that the group $G$ has an abelian elementary $p_r$-subgroup of rank at least $r$.

Suppose that for some natural number $r$ the centralizer of every abelian elementary subgroup of rank at least $r$ has finite rank.

Let $d$ be an arbitrary natural number. As was shown above, $G$ has an abelian elementary $q_0$-subgroup $A$ (for some prime number $q_0$) of rank at least $r + d$. From P. Hall' s theorems on Sylow bases of solvable groups, in view of the local solvability of $G$ and the finiteness of its Sylow $p$-subgroups, it follows that any Sylow $p$-subgroup (for any $p$) of the group $G$ is complemented in $G$. In particular, the Sylow $q_0$-subgroup $P$ containing $A$ is complemented. Let $T$ complement $P$ in $G$. Since $P$ is a finite group, $T$ has finite index in $G$. The intersection $N$ of all subgroups of $G$ conjugate to $T$ also has finite index in $G$. It is clear that $N$ has infinite rank. Put $Q_0 = AN$. Two cases are possible: either the centralizer of every element distinct from 1 of the group $A$ has finite rank, or the centralizer $Q_1$ of some element $a_1 \neq 1$ of the group $A$ has infinite rank. It is clear that $Q_1$ contains $A$ and is the product of the subgroups $A$ and $N_1 = N \cap Q_1$. The subgroup $N_1$ is an invariant subgroup of infinite rank in $Q_1$.

Now consider, instead of $Q_0$, its subgroup $Q_1$. Again two cases are possible: either the centralizer of every element of $A \setminus \{a_1\}$ in $Q_1$ has finite rank, or in $A \setminus \{a_1\}$ there is an element $a_2$ whose centralizer in $Q_1$ has infinite rank; denote this centralizer by $Q_2$. It is clear that $Q_2$ is the centralizer in $Q_0$ of the group $\{a_1, a_2\}$. Continuing this process further, we obtain, in view of the assumption that the rank of the centralizer of any subgroup of rank at least $r$ is finite, such a subgroup $\{a_1, a_2, \dots, a_k\}$, $k < r$, of the group $A$, that the rank of the centralizer $Q_k$ of the group $\{a_1, a_2, \dots, a_k\}$ is infinite, while the centralizer in $Q_k$ of any element of $A \setminus \{a_1, a_2, \dots, a_k\}$ has finite rank.

Let $A = \{a_1, a_2, \dots, a_k\} \times B$. Put $Q = BM$, where $M = N \cap Q_k$. As shown above, the centralizer in $Q$ of every element distinct from 1 of $B$ has finite rank.

Let $R'$ be a Sylow $p$-subgroup (for any $p$) of the group $M$. Since $R'$ is finite, by Sylow' s theorem $Q = M \cdot N(R')$, where $N(R')$ is the normalizer of the group $R'$ in $Q$. Therefore $N(R')$ contains a $q_0$-subgroup $B'$ conjugate to $B$. But this means that some Sylow $p$-subgroup $R$, conjugate to $R'$, is admissible with respect to $B$; in other words, $B$ is contained in the normalizer of $R$. Since here $p$ was chosen arbitrarily and since the ranks of the Sylow $p$-subgroups of the group $M$ are not bounded in the aggregate, $M$

has a set

$$P_1, P_2, \dots, P_n, \dots$$

of $B$-admissible $p_n$-subgroups whose ranks strictly increase. Since the rank of the centralizer of each element of the group $B$ distinct from 1 is finite, there exists a natural number $n_0$ such that, for $n \geq n_0$, $P_n$ is not contained in the centralizer of any element of $B$ distinct from 1. We may assume that $n_0 = 1$. Thus, $B$ is faithfully represented by automorphisms of the groups $P_n$. Let $D_n = D(P_n)$ be the Frattini subgroup of the group $P_n$. Then $B$ can be faithfully represented by automorphisms of the factor groups $P_n/D_n$. In view of Lemma 4, proved below, the ranks of the groups $P_n/D_n$ may be assumed to be increasing.

Let $b$ be an arbitrary element of $B$ distinct from 1, and let $S_n/D_n$ be the maximal subgroup of $P_n/D_n$ on which the element $b$ acts identically. Since the centralizer of the element $b$ has finite rank, the ranks of the groups $S_n/D_n$ are bounded by one and the same number $l(b)$, depending only on $b$. Let

$$P_n/D_n = P_n^{(1)}/D_n \times P_n^{(2)}/D_n \times \dots \times P_n^{(k_n)}/D_n$$

be the decomposition of $P_n/D_n$ into factors irreducible relative to $B$. If $h$ is the order of the group $B$ and $l = \max_{b \in B} l(b)$, then no more than $hl$ factors of the decomposition contain elements distinct from 1 that remain fixed under the transformation of the group $P_n/D_n$ by some element of the group $B$. Since the rank of each factor does not exceed $h$, and the ranks of $P_n/D_n$ increase, there exists such an $n$ that $k_n > hl$, and therefore in some factor $P_n^{(i)}/D_n$ the group $B$ induces a group of regular automorphisms. As is known, $B$ then has rank 1, which contradicts the assumption that the choice of the number $d$ was arbitrary. Thus, the assumption made at the beginning of the proof that the centralizer of an arbitrary abelian group of rank at least $r$ has finite rank is false.

Now, to complete the proof of Lemma 3, it remains only to prove the following lemma:

**Lemma 4.** *Let $G$ be a periodic locally soluble group with finite Sylow $p$-subgroups for all $p$. Then, if $G$ has infinite rank, and the centralizer of every abelian subgroup of $G$ of rank $r$ has finite rank, there exist in $G$ Sylow $p_i$-subgroups*

$$P_1, P_2, \dots, P_i, \dots,$$

*such that the ranks of their factor groups*

$$P_1/D_1, P_2/D_2, \dots, P_i/D_i, \dots$$

*are unbounded in the aggregate; here $D_i = D(P_i)$ is the Frattini subgroup of the group $P_i$.*

**Proof.** Just as in the proof of the preceding lemma, one can show that $G$ has abelian elementary $q$-subgroups $A_i$, $i = 1, 2, \dots$, whose ranks are $r + i$. Just as in the preceding lemma, one can show that in $G$ there exists a sequence of Sylow $A$-admissible $p_l$-subgroups

$$S_1, S_2, \dots, S_l, \dots,$$

whose ranks increase. Let $B_l$ be the largest subgroup of the group $A_i$ inducing the identity in $S_l$. If among the $B_l$ $(l = 1, 2, \dots)$ there are infinitely many such groups that the factor group $A_i/B_l$ has rank less than $i$, i.e. the $B_l$ have rank greater than $r$, then among these $B_l$, in view of the finiteness of the group $A_i$, one subgroup, for example $B_m$, is repeated infinitely many times. Therefore the centralizer $B_m$ has infinite rank, in contradiction

with the condition of the lemma. Thus, only a finite number of groups $A_i/B_i$ have rank less than $i$. Suppose that, for some $l_0$, $A_i/B_{l_0}$ has rank at least $i$. Then $A_i$ induces in $S_{l_0}$ an elementary abelian group of automorphisms of rank at least $i$, and therefore $S_{l_0}$ may be taken as the required group $P$. The lemma is proved.

We proceed to the proof of Theorem 1. Let $G$ be a periodic locally solvable group of infinite rank. If at least one abelian $p$-subgroup (for some $p$) has infinite rank, then the theorem is true. Suppose that the ranks of the abelian $p$-subgroups of $G$ (for all $p$) are finite. Then the Sylow $p$-subgroups of $G$ are special (see (6)), and the latter is an extension of a complete abelian group $A$ by a group with finite Sylow $p$-subgroups for all $p$ (see (7)). If $A$ has infinite rank, then the theorem is proved. Let the rank of $A$ be finite. Then the rank of $G/A$ is infinite. By Lemma 4 from (4), it is enough to show that $G/A$ has an abelian subgroup of infinite rank. Thus the general case has been reduced to the case of Lemma 3, and therefore the proof of Theorem 1 is complete.

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*Note: Figure translations are in progress. See original paper for figures.*

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