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Abstract

Full Text

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On a Boundary-Value Problem for a Vector Holomorphic in a Half-Space

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In this note we show that the Riemann–Hilbert boundary-value problem for a vector holomorphic in a half-space is Fredholm, if the vector of the boundary condition does not lie in the tangent plane. The definition of a holomorphic vector is given in the works ([1], [2]).

Consider the problem Γ:

Problem Γ. Find a vector V with components $p(x), u(x), v(x), w(x)$ (where $x(x_1, x_2, x_3)$ is a point of the three-dimensional Euclidean space E_3), holomorphic in the half-space $x_3 > 0$, vanishing at infinity, Hölder-continuous up to the boundary $x_3 = 0$, and satisfying on the plane $x_3 = 0$ the boundary conditions

$$\lambda_{11}p + \lambda_{12}u + \lambda_{13}v + \lambda_{14}w = f_1,$$

$$\lambda_{21}p + \lambda_{22}u + \lambda_{23}v + \lambda_{24}w = f_2, \tag{1}$$

where $\lambda_{ij}(z)$ and $f_i(z)$ $(i = 1, 2;\ j = 1, 2, 3, 4)$ are real functions of the point $z(x_1, x_2)$ of the plane $x_3 = 0$ of class $C_\beta(E_2)$, $0 < \beta < 1$, and the $f_i(z)$ satisfy the condition $f_i \in L_p^*$. The solution of the problem will naturally be sought in the class of functions whose boundary values belong to $L_p(E_2)$, $p > 2/\beta$.

Denote by Λ_{ij} the determinant composed of the i-th and j-th columns of the matrix of the boundary condition, and let

$$A(z) = \Lambda_{12} + \Lambda_{34}; \qquad B(z) = \Lambda_{13} + \Lambda_{42}; \qquad C(z) = \Lambda_{14} + \Lambda_{23}. \tag{2}$$

The vector with coordinates (A, B, C) will be called the **vector of the boundary condition**.

In the particular case when the matrix of the boundary condition has the form

$$\begin{Vmatrix} g_1 & g_2 & g_3 & g_4 \\ -g_4 & -g_3 & g_2 & g_1 \end{Vmatrix}$$

and the coefficients g_i $(i = 1, 2, 3, 4)$ are constant, this problem was considered by A. V. Bitsadze ([2]). It was assumed there that $C = g_1^2 + g_2^2 + g_3^2 + g_4^2 \neq 0$.

We shall assume that the condition

$$C(z) \neq 0 \tag{3}$$

is satisfied everywhere, including the point at infinity.

Theorem. *Under the assumptions made: 1) problem Γ is normally solvable; 2) the homogeneous problem and its adjoint have only a finite number of linearly independent solutions; 3) the index of problem Γ is equal to zero, i.e., the problem is Fredholm. (By the index we mean the index of the system of singular integral equations equivalent to problem Γ.)*

Condition (3) means that the vector of the boundary condition (2) does not lie in the tangent plane. For one second-order equation and a finite domain this result was obtained by Giraud. In one very special case

* The functions $\lambda_{ij}(z)$ have finite limits as $|z| \to \infty$.

it to systems of equations of the 2nd order ([3]). The system that we consider can, generally speaking, be reduced to a system of equations of the 2nd order; however, in the inverse reduction one has to solve a problem equivalent to the original one. Moreover, in order to apply Giraud' s results the half-space must be reduced to a bounded domain. In doing so, singularities appear in the coefficients of the original system.

We shall rely on the known integral representation of a vector holomorphic in a half-space (see ([2])):

$$V(x) = \frac{1}{2\pi} \iint_{E_2} D \frac{1}{|x-\xi|} \gamma^3 \mu(\xi)\, d\xi \tag{4}$$

(we use the notation from ([8])), where $\mu = (\mu_1, 0, 0, \mu_2)$ and μ_1 and μ_2 are the boundary values of the 1st and 4th components of $V(x)$ on the plane $x_3 = 0$.

Using the jump formulas for a two-dimensional integral of Cauchy type ([2]), we obtain a system of singular integral equations

$$H\mu = f, \tag{5}$$

where $\mu = (\mu_1, \mu_2)$, $f = (f_1, f_2)$, and the operator H has the form

$$H\mu \equiv L\mu + \frac{1}{2\pi} G \iint_{E_2} \hat{D} \frac{1}{|x-\xi|} \mu(\xi)\, d\xi.$$

The matrices L, G, and $\hat{D}$ are given by the formulas

$$L = \left\| \begin{matrix} \lambda_{11} & \lambda_{14} \\ \lambda_{21} & \lambda_{24} \end{matrix} \right\|, \qquad G = \left\| \begin{matrix} \lambda_{12} & \lambda_{13} \\ \lambda_{22} & \lambda_{23} \end{matrix} \right\|, \qquad \hat{D} = \left\| \begin{matrix} -\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} \end{matrix} \right\|.$$

By the uniqueness of representation (4), the original problem is completely equivalent to system (5).

We consider system (5) in the space $L_p(E_2)$, $p > 2/\beta$. Let $\Phi(z, y/|y|)$ be the symbol of the operator H. Then $\Phi = L + iGY$, where

$$Y = \left\| \begin{matrix} -\dfrac{y_2}{|y|} & \dfrac{y_1}{|y|} \\ \dfrac{y_1}{|y|} & \dfrac{y_2}{|y|} \end{matrix} \right\|, \qquad y = (y_1, y_2) \in E_2.$$

The symbol $\Phi(z, y/|y|)$, obviously, belongs to the class C_β^∞ introduced in ([4]), and by definition H is a singular operator of class C_β^∞.

Since $\det \Phi = C + iA\dfrac{y_1}{|y|} + iB\dfrac{y_2}{|y|}$ is nonzero by virtue of condition (3), there exists a matrix $\Phi' \in C_\beta^\infty$ such that $\Phi' \cdot \Phi = E$ (E is the identity matrix of the 2nd order).

Then, as is known, the operator $H' \in C_\beta^\infty$ corresponding to the matrix Φ' (see ([4])) is a regularizer for the operator H (see ([5])). Similarly it is shown that the adjoint operator H^* admits regularization. Assertions 1) and 2) of the theorem now follow from the theorems of § 2 of the book ([5]).

Let us show that the index $\chi(H)$ of the operator H is equal to zero. Let, for definiteness, $C(z) > 0$. Consider the system of singular integral equations $H_s\mu = f$, where

$$L_s = \left\| \begin{matrix} g_1 + s\lambda_1 & g_4 + s\lambda_1 \\ -g_4 + s\lambda_4 & g_1 - s\lambda_1 \end{matrix} \right\|; \qquad G_s = \left\| \begin{matrix} g_2 + s\lambda_2 & g_3 + s\lambda_3 \\ -g_3 + s\lambda_3 & g_2 - s\lambda_2 \end{matrix} \right\|; \qquad 0 \le s \le 1;$$

$$g_1 = \frac{\lambda_{11} + \lambda_{24}}{2}; \qquad g_2 = \frac{\lambda_{12} + \lambda_{23}}{2}; \qquad g_3 = \frac{\lambda_{13} - \lambda_{22}}{2}; \qquad g_4 = \frac{\lambda_{14} - \lambda_{21}}{2};$$

$$\lambda_1 = \frac{\lambda_{11} - \lambda_{24}}{2}; \qquad \lambda_2 = \frac{\lambda_{12} - \lambda_{23}}{2}; \qquad \lambda_3 = \frac{\lambda_{13} + \lambda_{22}}{2}; \qquad \lambda_4 = \frac{\lambda_{14} + \lambda_{21}}{2}.$$

Let Φ_s be the symbol of the operator H_s. Then

$$\det \Phi_s = C + (1 - s^2)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) > 0$$

and, obviously, $H_s \in C_\beta^\infty$.

The elements of the symbolic matrix $\Phi_{s+\Delta s} - \Phi_s$ and their derivatives up to the 4th order with respect to the coordinates of the point y, (y_1, y_2), as is easy to see, are small together with Δs for $|y| \geqslant 1$, and, by Theorem 3 of paper ([4]), the norm $\|H_{s+\Delta s} - H_s\|_p$ of the operator $H_{s+\Delta s} - H_s$ in the space L_p, $p > 2/\beta$, is small together with Δs.

By a known theorem (see ([5]), §2), $\varkappa(H_{s+\Delta s}) = \varkappa(H_s)$, i.e. the index of the operator H_s does not depend on s. But for $s = 1$ we have the operator H, and for $s = 0$ the operator H_0. Consequently,

$$\varkappa(H) = \varkappa(H_0).$$

This equality can also be obtained from a theorem of B. V. Boyarskii (see ([6]), appendix to Ch. II). The symbolic determinant of the operator H_0 is

$$\det \Phi_0 = g_1^2 + g_2^2 + g_3^2 + g_4^2 = \rho^2 > 0.$$

Without loss of generality we may suppose that $\rho(z) \equiv 1$. If this is not so, consider the operator $P\mu = \dfrac{1}{\rho}\mu$. Let $H_0' = PH_0$. Obviously, $H_0' \in C_\beta^\infty$ and $\varkappa(P) = 0$. Passing to the symbolic matrices, we have

$$\Phi_0' = \frac{1}{\rho} E \cdot \Phi_0$$

and $\det \Phi_0' = 1$. In this case (see ([5]), §2)

$$\varkappa(H_0') = \varkappa(P) + \varkappa(H_0) = \varkappa(H_0).$$

Consider now the system

$$H_0 \mu = f, \tag{6}$$

assuming that

$$g_1^2 + g_2^2 + g_3^2 + g_4^2 = 1. \tag{7}$$

By virtue of condition (7), there exist angles $\theta(z), \varphi(z)$, and $\psi(z)$, $0 \leqslant \theta(z) \leqslant \pi$; $0 \leqslant \varphi(z) \leqslant \pi$; $0 \leqslant \psi(z) < 2\pi$, such that

$$g_1 = \cos\theta; \quad g_2 = \sin\theta\cos\varphi; \quad g_3 = \sin\theta\sin\varphi\cos\psi;$$

$$g_4 = \sin\theta\sin\varphi\sin\psi. \tag{8}$$

Let $H_0^{(s)}\mu = f$ be a system of the form (6), but with coefficients $g_i^{(s)}(z)$, $(i = 1, 2, 3, 4)$, defined by formulas (8) with the angles $s\theta(z), \varphi(z), \psi(z)$, where $0 \leqslant$

$s \leqslant 1$. The symbolic determinant of this system is identically equal to one. Similarly to what was done above, it is proved that $\varkappa(H_0^{(s)})$ does not depend on s. Consequently, the index of the system (6) coincides with the index of the system $\mu = f$, i.e. is equal to zero. Therefore the index of the original system (5) is equal to zero in the space $L_p(E_2),\ p > 2/\beta$.

Let μ_0 be a solution of the homogeneous system (5), and let the operator $H' \in C_\beta^\infty$ be the regularizer indicated above for the operator H, i.e. $H'H = I + T$, where I is the identity operator and T is a completely continuous operator in the space $L_p,\ p > 2/\beta$. Then μ_0 satisfies the equation $\mu_0 = -T\mu_0$.

It is easy to see that T is an integral operator with a weak singularity. From the properties of the operator T (see [7]) it follows that $\mu_0 \in C_\delta(E_2)$, $\delta = \dfrac{p\beta - 2}{p}$, and, consequently, $\varkappa = 0$ in $C_\delta L_p(E_2)$, i.e. the problem Γ is Fredholm.

If the coefficients of the boundary condition (1), λ_{ij}, are constant, then the problem Γ is always solvable, since, as the Fourier transform shows, the homogeneous problem Γ_0 has only the trivial solution.

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