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Abstract

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BEHAVIOR OF SPECTRAL FUNCTIONS OF DIFFERENTIAL SYSTEMS WITH BOUNDARY CONDITIONS AT A SINGULAR END

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1. Let $M(x)$ be a nondecreasing finite function defined on the interval (a, b) $(-\infty \leq a < b \leq +\infty)$. The left end $x = a$ of this interval will be called **regular** if the set of values of the function $M(x)$ and the set of its points of increase are bounded below (the latter is certainly satisfied if $a > -\infty$). Otherwise the end $x = a$ will be called **singular**. Analogous definitions may be introduced for the end $x = b$.

We shall regard a function $\varphi(x)$ as a solution of the generalized differential equation $\frac{d}{dM} y^-(x) = g(x)$* if it is absolutely continuous on (a, b), at each point $x \in (a, b)$ has a left derivative $\varphi^-(x)$, which in turn is M-absolutely continuous, and the equality $\frac{d}{dM}\varphi^-(x) = g(x)$ holds M-almost everywhere on (a, b).

We shall assign a nondecreasing function $M(x)$ to the class $\mathfrak{M}$ if the interval (a, b) on which it is defined has an infinite left end $a = -\infty$ and $M(x) \in \mathcal{L}_1(-\infty, c)$, where $-\infty < c < b$. We note that if $M(x) \in \mathfrak{M}$, then the mass distribution generated by the function $M(x)$ on $(-\infty, c)$ has a finite static moment with respect to the point c $(-\infty < c < b)$. Conversely, from the fulfillment of the latter condition it follows that the function $M(x)$ can be normalized by means of an additive constant so that it belongs to $\mathfrak{M}$.

If $M(x) \in \mathfrak{M}$, then the differential system

$$-\frac{d}{dM} y^-(x) - \lambda y(x) = 0 \quad (-\infty > x > b), \quad \lim_{x \to -\infty} y(x) = 1 \tag{1}$$

has at least one spectral function** (even in the case when the left–

* Recently the string operator $\left(Sy = -\frac{d}{dM} y^-(x)\right)$ with an arbitrary mass distribution has begun to play a role in the theory of Markov processes. Operators in such a notation occur in the author' s paper ([3]) and in his dissertation ([5]). In another form these operators were studied by M. G. Krein already in ([1]), and also in ([8–13]), where for the first time a theory of spectral functions was constructed for the case of a string with one or two regular ends. Through a misunderstanding, K. Itô ([2]) calls these operators Feller operators, although W.

Feller began to study them later ([15]), and even then excluded from consideration the case in which the function $M(x)$ has intervals of constancy.

** We adhere to the definition of a spectral function given in ([3]). Let us recall it as applied to system (1). Let $u(x;\lambda)$ be the (unique) solution of differential system (1). A nondecreasing function $\tau(\lambda)$ $(-\infty < \lambda < \infty;\ \tau(\lambda - 0) = \tau(\lambda);\ \tau(0) = 0)$ is called a spectral function of this differential system if the mapping $U_\tau : f \to F$, where $f(x) \in \mathcal{L}^2_M(-\infty, b)$, and

$$F(\lambda) = \int_{-\infty}^{b} f(x)u(x;\lambda)\,dM(x) \tag{*}$$

isometrically maps the Hilbert space $\mathcal{L}^2_M(-\infty, b)$ into the Hilbert space $\mathcal{L}^2_\tau(-\infty, \infty)$ (the convergence of the integral in $(*)$ is understood in the sense of the metric in $\mathcal{L}^2_\tau(-\infty, \infty)$).

left end $x = -\infty$ is singular, and at it there occurs the case of a Weyl point, independently of the behavior of $M(x)$ in a left neighborhood of the point $x = b$. The latter assertion follows from our Theorem 3 in ([3]) (see also ([5])) as a special case. Among the spectral functions $\tau(\lambda)$ of the differential system (1) there are **positive** ones, i.e. such that $\tau(\lambda) = 0$ for all $\lambda = 0$.

Soon after the existence of spectral functions of the differential system (1) for $M(x) \in \mathfrak{M}$ had been proved, M. G. Krein posed the problem: to find the set T of all nondecreasing functions $\tau(\lambda)$ on $(-\infty, \infty)$ for which there exists a function $M(x) \in \mathfrak{M}$ such that $\tau(\lambda)$ is a spectral function of system (1). This problem plays a large role in the theory of extrapolation and filtering of random processes (see ([12])). Moreover, it plays a large role in the question of the possibility of constructing, for a given weight, a natural generalized Fourier transform.

Since in the case when the left end $x = -\infty$ is regular one can normalize the function $M(x)$, by means of an additive constant, so that it belongs to $\mathfrak{M}$, the well-known theorem of M. G. Krein (([9]), Theorem 3), giving conditions necessary and sufficient for a nondecreasing function $\tau(\lambda)$ to be a positive spectral function of a string with a regular left end, shows that every nondecreasing function $\tau(\lambda)$ $(-\infty < \lambda < \infty;\ \tau(\lambda - 0) = \tau(\lambda))$ such that $\tau(\lambda) = 0$ for $\lambda < 0$ and $\lambda^{-2}\tau(\lambda) \in L_1(1, \infty)$, belongs to T. After the appearance of the paper ([14]) of M. G. Krein it became clear that the set T contains every nondecreasing function $\tau(\lambda)$ $(-\infty < \lambda < \infty;\ \tau(\lambda - 0) = \tau(\lambda))$, equal to zero for $\lambda < 0$, and, as $\lambda \to +\infty$, increasing no faster than a power function.

Naturally, the question arose whether there are functions in T which on $(2, \infty)$ are not majorized by any power function. The following Theorem 1, connecting the behavior of the function $M(x)$ as $x \to -\infty$ with the growth, as $\lambda \to +\infty$, of the spectral functions $\tau(\lambda)$ of system (1), gives, in particular, an affirmative answer to this question.

Theorem 1. Let $M(x) \in \mathfrak{M}$ and let the end $x = -\infty$ be singular. Then from the fact that, for some real α, the inequality

$$\int_{-\infty}^{c} \left(\int_{-\infty}^{x} M(s)\, ds \right)^{\alpha - 1} dx < \infty \qquad (-\infty < c < b)^{*}, \tag{2}$$

holds, it follows that for every spectral function $\tau(\lambda)$ of system (1) the inequality

$$\int_{1}^{\infty} \frac{\tau(\lambda)}{\lambda^{\alpha+1}}\, d\lambda < \infty \tag{3}$$

holds.

Conversely, if even for one positive spectral function $\tau(\lambda)$ of system (1) (3) holds, then (2) holds.

In the course of the proof of Theorem 1 we generalized some results of M. G. Krein (Theorem 2 from ([10]) and Theorem 2 from ([11])), which essentially concern differential system (1) with a regular left end, to the case of a singular left end. Thanks to this it was possible to carry over our results ([4–6]) on the behavior of spectral functions in the case of regularity of the end, at which the boundary condition is prescribed, to the case considered here.

Let us note that if one puts, for example, $M(x) = |x|^{-1} \log^{-2} |x|$ $(-\infty < x < b = -1)$, then inequality (2) will not hold for any real α; consequently, for a positive spectral function $\tau(\lambda)$ of system (1), inequality (3) does not hold for any real α, i.e. the function $\tau(\lambda)$ is not majorized on $(2, +\infty)$ by a power function.

* If the end $x = -\infty$ is singular, inequality (2) and, consequently, (3) for positive

spectral functions $\tau(\lambda)$ cannot hold for $\alpha \leqslant 1$. In the case when this end is regular, inequality (3), as M. G. Krein showed, holds for $\alpha = 1$.

2. The following theorem, which gives a description of the set T, made it possible to understand better what growth as $\lambda \to +\infty$ is admissible for functions $\tau(\lambda) \in T$.

Theorem 2. Take two arbitrary interlacing increasing sequences (not necessarily infinite) of positive numbers $\{\lambda_j\}$ and $\{\mu_j\}$ such that $\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots$ and $\sum \lambda_j^{-1} < \infty$, an arbitrary positive constant h, and an arbitrary R-function* $\omega(z)$ or the function $\omega(z) \equiv \infty$. Construct the functions

$$D(z) = \prod_j \left(1 - \frac{z}{\lambda_j}\right), \qquad E(z) = -hz \prod_j \left(1 - \frac{z}{\mu_j}\right), \tag{4}$$

$$\Omega(z) = \left(D^2(z) + E^2(z)\right)^{-1} (D(z)\omega(z) - E(z))(E(z)\omega(z) + D(z))^{-1}.$$

$$(\operatorname{Im} z \neq 0).$$

Then the function $\tau(\lambda)$, defined by the equality

$$\tau(\lambda)=\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_0^\lambda \operatorname{Im}\Omega(s+i\varepsilon)\,ds \qquad (-\infty<\lambda<\infty),$$

belongs to T. By this construction one can obtain any function $\tau(\lambda)\in T$.

This theorem is ineffective in the sense that, for a given function $\tau(\lambda)$, it does not allow one to judge whether it belongs to the set T. At the same time, from it there follows a number of important sufficient conditions and some necessary conditions for membership in this set.

Corollary 1. If $D(z)$ and $E(z)$ are entire functions defined by equalities (4), where $0<\lambda_1<\mu_1<\lambda_2<\mu_2<\cdots$, $h>0$, then the function $\tau(\lambda)$, equal to zero for $\lambda<0$, and for $\lambda\geq 0$ having derivative equal to $\pi^{-1}\lambda^{1/2}(\lambda D^2(\lambda)+E^2(\lambda))^{-1}$, belongs to T.

This is obtained from Theorem 2 if one sets $\omega(z)=(-z)^{-1/2}$ for $z\in[0,+\infty)$ (here one considers that branch of the function $(-z)^{1/2}$ which takes positive values for $z<0$).**

Corollary 2. For whatever $\alpha\in(1/2,1)$, in the set T there are functions growing, as $\lambda\to+\infty$, no more slowly than $\exp\lambda^\alpha$.

This follows from Corollary 1 and the well-known theorems on the behavior of entire functions with positive zeros, if one assumes $\lambda_n=n^{1/\alpha}$ $(n=1,2,3,\ldots)$.

Let $\{\nu_j\}_1^\infty$ be an arbitrary sequence of positive numbers such that $\sum j^{-2}\nu_j^{-1}<\infty$. If we set $E(z)=-(\nu_0\pi)^{-1}\sqrt{z}\sin(\pi\sqrt{z})$, define the function $D(z)$ from the equality

$$D(z)=E(z)\left(-\frac{\nu_0}{z}+\sum_{j=1}^{\infty}\frac{4\nu_0}{\nu_j(j^2-z)}\right),$$

and take $\omega(z)\equiv\infty$, then it is easy to verify that the function $\tau(\lambda)$ obtained by means of the construction described in Theorem 2 will have no other points of growth except the points $\mu_j=j^2$, at which its jumps will be equal to ν_j $(j=0,1,2,3,\ldots)$. Hence follows

Corollary 3. In T there are functions growing arbitrarily rapidly as $\lambda\to+\infty$.

At the same time, not every nondecreasing function $\tau(\lambda)$ on $(-\infty,\infty)$ belongs to T, even if it is identically equal to zero for $\lambda<0$. This follows from the following proposition.

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* Here, as in a number of our other works (5–7), we call a function $\omega(z)$ an R-function if it is defined and holomorphic in each of the half-planes $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$, and: a) $\omega(\bar{z}) = \overline{\omega(z)}$, b) $\operatorname{Im}\omega(z) \cdot \operatorname{Im} z \geq 0$ $(\operatorname{Im} z \neq 0)$.

** By taking other R-functions as $\omega(z)$, one can obtain a series of analogous sufficient conditions for membership in the set T.

Theorem 3. If $\tau(\lambda) \in T$, then there exist a function $\xi(\lambda)$, positive on $(1, \infty)$ and tending to zero as $\lambda \to +\infty$, and a system of nonintersecting intervals (α_j, β_j) $(j = 1, 2, \ldots;\ \alpha_{j+1} > \alpha_j)$ with arbitrarily small sum of lengths, such that

$$\int_G \exp(-\lambda\xi(\lambda)\log\lambda)\, d\tau(\lambda) < \infty,$$

where G is the set of points of the interval $(1, \infty)$ that do not belong to any of the intervals (α_j, β_j) $(j = 1, 2, \ldots)$.

The assertion of this theorem follows easily from Theorem 2, if one takes into account that the function $\Omega(z)(D^2(z) + E^2(z))$ is an R-function.

3. In conclusion we present several propositions relating the behavior of the spectral function $\tau(\lambda)$ of system (1) as $\lambda \to +\infty$ to the behavior of the function $M(x)$ as $x \to -\infty$.

Theorem 4. If, in some right neighborhood of the point $x = -\infty$, the function $M(x) > 0$ is majorized by a function $M_0(x) \in \mathfrak{M}$ (and hence $M(x) \in \mathfrak{M}$) such that $|x|M_0(x)$ increases in this neighborhood, then for any spectral function $\tau(\lambda)$ of system (1), as $\lambda \to +\infty$ one has the asymptotic equality

$$\log\tau(\lambda) = o(\lambda).$$

It is clear that in the preceding theorem, as $M_0(x)$ one may take one of the functions

$$M_{1,\alpha}(x) = |x|^{-1}(\log|x|)^{-\alpha}, \quad M_{2,\alpha}(x) = |x|^{-1}(\log|x|)^{-1}(\log_2|x|)^{-\alpha},$$

$$M_{3,\alpha}(x) = |x|^{-1}(\log|x|)^{-1}(\log_2|x|)^{-1}(\log_3|x|)^{-\alpha}, \ldots,$$

where $\alpha > 1$*. With this choice of $M_0(x)$ the assertion of Theorem 4 can be sharpened; namely, the following is true.

Theorem 5. If, as $x \to -\infty$, the asymptotic equality

$$M(x) = O(M_{1,\alpha}(x)),$$

where $\alpha > 1$, holds, then for any spectral function $\tau(\lambda)$ of system (1), as $\lambda \to +\infty$ one has the asymptotic equality

$$\log\tau(\lambda) = O(\lambda^{1/\alpha}).$$

If, however, $M(x) = O(M_{n,\alpha}(x))$ as $x \to -\infty$, where n is a natural number different from one and $\alpha > 1$, then

$$\log \tau(\lambda) = O(\lambda(\log_{n-1} \lambda)^{1-\alpha})$$

as $\lambda \to +\infty$.

It has been possible to show that if $M(x) = M_{n,\alpha}(x)$ $(-\infty < x < b)$, where $n > 1$, $\alpha > 1$, then any spectral function $\tau(\lambda)$ of system (1) is such that, in some left neighborhood of the point $\lambda = +\infty$, the ratio

$$\log \tau(\lambda)/\lambda(\log_{n-1} \lambda)^{1-\alpha}$$

is enclosed between two positive constants. In this sense, the assertion of Theorem 5 for $n > 1$ is sharp. In the case when $n = 1$, we have been able to establish the sharpness of the assertion of Theorem 5 only for $\alpha = 2$. This is obtained as a result of the analysis of an example considered by E. Hille in his paper (16).

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* Here $\log_2 m = \log\log m, \quad \log_{n+1} m = \log\log_n m$ $(n = 2, 3, 4, ...)$.

Note: Figure translations are in progress. See original paper for figures.