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Abstract

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MATHEMATICS

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SOLUTION OF A MULTIDIMENSIONAL KINETIC EQUATION BY THE SPLITTING METHOD

Consider the one-speed equation describing neutron transport in a medium with isotropic scattering:

$$\frac{1}{v}\frac{\partial\varphi}{\partial t}+\vec{\Omega}\nabla\varphi+\sigma\varphi=\frac{\sigma_s}{4\pi}\int d\vec{\Omega}'\varphi(\mathbf{r},\vec{\Omega}',t)+S(\mathbf{r},\vec{\Omega},t). \tag{1}$$

For equation (1) we pose the mixed Cauchy problem

$$\varphi(\mathbf{r},\vec{\Omega},t)=0,\qquad \vec{\Omega}\mathbf{n}<0,\quad \mathbf{r},\vec{\Omega},t\in\Gamma; \tag{2}$$

$$\varphi(\mathbf{r},\vec{\Omega},0)=\Phi(\mathbf{r},\vec{\Omega}),\qquad \mathbf{r},\vec{\Omega}\in D, \tag{3}$$

in the cylindrical domain $\Pi = D\times T$ with base D, lateral boundary $\Gamma=\gamma\times T$, $\gamma=\overline{D}-D$.

Along with the nonstationary problem (1)–(3), we shall consider the stationary problem (cf. (1))

$$\vec{\Omega}\nabla\varphi+\sigma\varphi=\frac{\sigma_s}{4\pi}\int d\vec{\Omega}'\varphi(\mathbf{r},\vec{\Omega}')+S(\mathbf{r},\vec{\Omega}), \tag{1a}$$

$$\varphi(\mathbf{r},\vec{\Omega})=0,\qquad \vec{\Omega}\mathbf{n}<0,\quad \mathbf{r},\vec{\Omega}\in\gamma. \tag{2a}$$

For integrating problem (1)–(3), let us apply the following scheme (the scheme of incomplete splitting):

$$\frac{\varphi^{n+1/2}-\varphi^n}{\tau}=\Lambda_1(\alpha\varphi^{n+1/2}+\beta\varphi^n)+\overline{S}, \tag{4a}$$

$$\frac{\varphi^{n+1}-\varphi^{n+1/2}}{\tau}=\Lambda_2(\alpha\varphi^{n+1}+\beta\varphi^{n+1/2}), \tag{4b}$$

where

$$\Lambda_1=-\sigma E+\frac{\sigma_s}{4\pi}\Delta\vec{\Omega}'\Sigma, \qquad \Lambda_2=-\vec{\Omega}\delta\varphi$$

are difference approximations of the operators

$$-\sigma E+\frac{\sigma_s}{4\pi}\int d\vec{\Omega}', \qquad -\vec{\Omega}\nabla\varphi,$$

respectively; $\overline{S}$ is an approximation of S, $\tau=v\Delta t$, $\alpha\geq 0$, $\beta\geq 0$, $\alpha+\beta=1$.

Scheme (4) is implemented as follows. Summing (4a) over all angles $\vec{\Omega}$, we obtain:

$$\frac{\varphi_0^{n+1/2}-\varphi_0^n}{\tau}=-\sigma_c(\alpha\varphi_0^{n+1/2}+\beta\varphi_0^n)+\overline{S}_0; \tag{5}$$

$$\varphi_0=\sum\varphi\Delta\vec{\Omega}', \qquad \overline{S}_0=\sum\overline{S}\Delta\vec{\Omega}', \qquad \sigma_c=\sigma-\sigma_s \tag{5a}$$

and, consequently,

$$\varphi_0^{n+1/2}=\frac{1-\beta\tau\sigma_c}{1+\alpha\tau\sigma_c}\varphi_0^n+\overline{S}_0\tau\cdot\frac{1}{1+\alpha\tau\sigma_c}. \tag{6}$$

After this, equation (4a) is integrated; it can be rewritten in the form

$$\frac{\varphi^{n+1/2}-\varphi^n}{\tau}+\sigma(\alpha\varphi^{n+1/2}+\beta\varphi^n)=\frac{\sigma_s}{4\pi}(\alpha\varphi_0^{n+1/2}+\beta\varphi_0^n)+\overline{S}. \tag{4c}$$

By a method analogous to (2), one can show that the scheme (4) of incomplete splitting converges for any τ/h, if $\alpha=1$.

Splitting the operator of the second fractional step, we obtain the **scheme of complete splitting**:

$$\frac{\varphi^{n+1/m}-\varphi^n}{\tau}=\Lambda_1(\alpha\varphi^{n+1/m}+\beta\varphi^n)+\overline{S}; \tag{7a}$$

Fig. 1

Figure 1: Fig. 1

$$\frac{\varphi^{n+(s+1)/m} - \varphi^{n+s/m}}{\tau} = \Lambda_{2s}(\alpha\varphi^{n+(s+1)/m} + \beta\varphi^{n+s/m}), \quad s = 1, 2, \dots, m-1, \tag{7b}$$

where $\Lambda_2 = \Lambda_{21} + \dots + \Lambda_{2m-1}$; Λ_{2i} are approximations of the one-dimensional operators $\Omega_i \partial/\partial x_i$, $i = 1, \dots, m-1$; $(m-1)$ is the dimension of space. Scheme (7) also converges for any τ/h, if $\alpha \geqslant 1/2$.

If the domain Π of problem (1)–(3) is a parallelepiped, the implementation of the boundary conditions (2) in the complete splitting scheme is obvious: at each s-th fractional step (7b) an upwind scheme is carried out along x_s in the direction from the illuminated face $(\overline{\Omega}n < 0)$. In the case of an arbitrary cylindrical domain Π_0 with base D_0, the latter is enclosed in a parallelepiped Π, and the values of φ are correspondingly completed in the domain $\Pi - \Pi_0$.

Fig. 1

Figure 1 shows how, in the two-dimensional case, φ is completed in the section $D = ABCD$ of the parallelepiped D at the time $t = (n + 1/3)\tau$.

When carrying out the first fractional step $(s = 1)$ from (7b), one sets:

$$\varphi^{n+2/3} = 0 \text{ on } AB; \qquad \varphi^{n+1/3} = 0 \text{ in } FBg,\ gAE.$$

After this, the second fractional step $(s = 2)$ from (7b) is performed in the square $ABCD$ with initial data $\varphi^{n+2/3}$, obtained in the indicated way, and with boundary conditions φ^{n+1} on AD. For $\alpha = \beta = 1/2$, schemes (4), (7) have second order of accuracy and may be used as relaxation schemes (iterative schemes) for solving the stationary problem (1a), (2a). In this case the scheme (4) of incomplete splitting will converge for any τ, while scheme (7) requires, for convergence, that the condition $\tau \to 0$ be satisfied.

Let us note that the iterative scheme (4) for relaxation can be realized as scheme (7) with internal iterations in (7b).

In an analogous way, in the case of incomplete splitting, a solution of problem (1a), (2a) can be obtained with the aid of the equation

$$(E - \tau\Lambda_1)(E - \tau\Lambda_2)\varphi^{n+1} = (E + \tau\Lambda_1)(E + \tau\Lambda_2)\varphi^n - 2\tau\overline{S}. \tag{8}$$

This scheme converges for any step τ.

The iterative relaxation scheme for $m > 2$ will have second order of accuracy in τ, if it is defined in the following way:

$$(E - \tau\Lambda_1)\prod_{s=1}^{m-1}(E - \tau\Lambda_{2s})\varphi^{n+1} = (E + \tau\Lambda_1)\prod_{s=1}^{m-1}(E + \tau\Lambda_{2s})\varphi^{n} - 2\tau\overline{S}. \quad (8a)$$

Splitting schemes are also expedient to apply in the one-dimensional case. The algorithms set out above are, in an obvious way, transferred to the case of an arbitrary scattering indicatrix and to problems with energy dependence.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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