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F. S. Khadzhimullaev

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Abstract

Full Text

F. S. Khadzhimullaev

ON A THEOREM OF D. STANCU

(Presented by Academician A. P. Kolmogorov on 28 III 1964)

In paper (^{1}), D. Stancu asserts the validity of the following theorem.

If the function $f(x, y)$ has bounded partial derivatives of order N $(N \geqslant 2)$ on $\Delta = \{x \geqslant 0,\ y \geqslant 0,\ x + y \leqslant 1\}$, then the asymptotic equality

$$B_n(f; x, y) = f(x, y) + \sum_{\nu=1}^{N} \frac{1}{n^\nu} \sum_{k=0}^{\nu} \frac{S^{(n)}_{\nu-k,k}(x, y)}{(\nu-k)!\,k!} f^{(\nu)}_{x^{\nu-k}y^k}(x, y) + \frac{\varepsilon_n}{n^s}, \qquad (1)$$

holds, where

$$B_n(f; x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} f\left(\frac{i}{n}, \frac{j}{n}\right) \binom{n}{i} \binom{n-i}{j} x^i y^j (1 - x - y)^{n-i-j}$$

is a Bernstein-type polynomial of degree n,

$$S^{(n)}_{\nu-k,k}(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} (i - nx)^{\nu-k} (j - ny)^k \binom{n}{i} \binom{n-i}{j} x^i y^j (1 - x - y)^{n-i-j},$$

$$s = \left[\frac{N+1}{2}\right], \quad \text{and } \varepsilon_n \to 0 \text{ as } n \to \infty.$$

Below we show the invalidity of this assertion and propose another theorem in its place.

The invalidity of Stancu' s theorem is easiest to detect in the case $N = 2$. In this case, as the author himself notes, formula (1) reduces to

$$B_n(f; x, y) = f(x, y) + \frac{x(1-x)}{2n} f''_{x^2}(x, y) + \frac{xy}{n} f''_{xy}(x, y) + \frac{y(1-y)}{2n} f''_{y^2}(x, y) + \frac{\varepsilon_n}{n}, \qquad (2)$$

which is a generalization to two variables of E. V. Voronovskaya' s formula.

Take, for example, the function

$$f(x,y)=\begin{cases}(x-\frac{1}{2})(y-\frac{1}{2})\left[\dfrac{(x-\frac{1}{2})^2-(y-\frac{1}{2})^2}{(x-\frac{1}{2})^2+(y-\frac{1}{2})^2}\right]^2, & \text{if } (x,y)\neq(\frac{1}{2},\frac{1}{2}),\\ 0, & \text{if } (x,y)=(\frac{1}{2},\frac{1}{2}).\end{cases}$$

It is not difficult to verify that this function has bounded partial derivatives of the second order on the whole plane and that for it, at the point $(\frac{1}{2},\frac{1}{2})$, relation (2) does not hold (one obtains $\varepsilon_n=\frac{1}{4}$).

Using the local Taylor formula from paper (^{2}) and applying a device analogous to the case of a function of one argument, one can prove the following theorem:

Theorem. If a function $f(x,y)$ admits on

$$\Delta := \{x\geq 0,\ y\geq 0,\ x+y\leq 1\}$$

bounded partial derivatives

$$f^{(N)}_{x^N},\ f^{(N)}_{x^{N-1}y},\dots,\ f^{(N)}_{xy^{N-1}},\ f^{(N)}_{y^N},$$

where $N\geq 2$, and, in addition, the derivative $f^{(N-1)}_{x^{N-1}}$ is differentiable at every point $(x,y)\in\Delta$, then the following asymptotic equality holds:

$$B_n(f;x,y)=f(x,y)+\sum_{\nu=1}^{N}\frac{1}{n^\nu}\sum_{k=0}^{\nu}\frac{S^{(n)}_{\nu-k,k}(x,y)}{(\nu-k)!\,k!}\,f^{(\nu)}_{x^{\nu-k}y^k}(x,y)+\frac{\varepsilon_n}{n^s}, \qquad (3)$$

where $B_n(f;x,y)$, $S^{(n)}_{\nu-k,k}(x,y)$, and ε_n denote the same quantities as in formula (1), and $s=N/2$.

An analogous theorem can also be proved in the case when the variables x and y interchange roles.

We note that, for these theorems to be valid, the existence of all derivatives of order N is by no means necessary, but only of certain ones; namely, when $f^{(N-1)}_{x^{N-1}}$ is differentiable, the derivatives

$$f^{(N)}_{x^N},\quad f^{(N)}_{x^{N-1}y},\dots,\quad f^{(N)}_{xy^{N-1}},\quad f^{(N)}_{y^N}$$

must exist; when $f^{(N-1)}_{y^{N-1}}$ is differentiable, the derivatives

$$f^{(N)}_{y^N},\quad f^{(N)}_{y^{N-1}x},\dots,\quad f^{(N)}_{yx^{N-1}},\quad f^{(N)}_{x^N}$$

must exist.

In the example given, neither of the derivatives f'_x and f'_y is differentiable at the point (1/2, 1/2).

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For formula (3) to be valid, the indicated conditions are apparently necessary.

In conclusion, the author expresses his gratitude to S. Kh. Sirazhdinov for his attention to this work.

Tashkent State University
named after V. I. Lenin

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[1] D. Stancu, *DAN*, **134**, No. 1 (1960). [2] F. S. Khadzhimullaev, *Scientific Proceedings of Tashkent State University named after V. I. Lenin*, issue 228 (1963).

Note: Figure translations are in progress. See original paper for figures.

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