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Abstract

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INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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1. Let D be a bounded domain with smooth boundary ∂D (i.e., ∂D is a differentiable manifold) in the space C^n of complex variables $(z_1, z_2, \dots, z_n) = z$. Denote $\bar{z} = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n)$, $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_n)$. Consider continuous functions $\Phi(z, \bar{z}, \xi, \bar{\xi})$ and $\varphi(z, \bar{z}, \xi)$, defined on the set * $\{(z, \xi) : z \in D,\ \xi \in \partial D\} \subset C^{2n}$ and satisfying the following conditions:

1) $\Phi(z, \bar{z}, \xi, \bar{\xi}) \neq 0$;

2) $\Phi(z, \bar{z}, \xi, \bar{\xi})$ can be represented in the form

$$\Phi = (\xi_1 - z_1)P_1 + (\xi_2 - z_2)P_2 + \dots + (\xi_n - z_n)P_n,$$

where $P_i = P_i(z, \bar{z}, \xi, \bar{\xi})$ have continuous derivatives with respect to the variables $(\xi, \bar{\xi})$, $i = 1, 2, \dots, n$;

3) $\varphi(z, \bar{z}, \xi)$ is holomorphic in ξ, where $\xi \in \partial D$ for $n > 1$ and $\xi \in \bar{D}$ for $n = 1$;

4) $\varphi(z, \bar{z}, z) \equiv 1$.

The main role in the present paper is played by the exterior differential form

$$\mu = \frac{(n-1)!\,\varphi}{(2\pi i)^n \Phi^n} \left(\sum_{k=1}^{n} \delta_k \, d\bar{\xi}_1 \wedge \dots \wedge d\bar{\xi}_{k-1} \wedge d\bar{\xi}_{k+1} \wedge \dots \wedge d\bar{\xi}_n \right) \wedge d\xi_1 \wedge \dots \wedge d\xi_n, \quad (1)$$

where $\wedge$ is the sign of exterior multiplication,

$$\delta_k = \begin{vmatrix} P_1 & P_2 & \cdots & P_n \\ P'_{1\bar{\xi}_1} & P'_{2\bar{\xi}_1} & \cdots & P'_{n\bar{\xi}_1} \\ \vdots & \vdots & & \vdots \\ P'_{1\bar{\xi}_{k-1}} & P'_{2\bar{\xi}_{k-1}} & \cdots & P'_{n\bar{\xi}_{k-1}} \\ P'_{1\bar{\xi}_{k+1}} & P'_{2\bar{\xi}_{k+1}} & \cdots & P'_{n\bar{\xi}_{k+1}} \\ \vdots & \vdots & & \vdots \\ P'_{1\bar{\xi}_n} & P'_{2\bar{\xi}_n} & \cdots & P'_{n\bar{\xi}_n} \end{vmatrix}.$$

Theorem. If the function $f(z)$ is holomorphic in the domain D and continuous in the closed domain $\bar{D}$, then for points $z \in D$ the formula

$$f(z) = \int_{\partial D} f(\xi)\, \mu \tag{2}$$

holds.

Proof consists of two steps:

a) Formula (2) is derived under the assumption that $\varphi(z, \bar{z}, \xi) \equiv 1$. This derivation differs in no essential way from the proof of the integral formula in the work (^{1}).

b) Let the function $\varphi(z, \bar{z}, \zeta)$ satisfy conditions 3) and 4). Then, using formula (2) under the assumptions of the first step, we obtain

$$\int_{\partial D} f(\zeta)\mu = f(z)\varphi(z, \bar{z}, z) = f(z).$$

2. All integral representations known to us for holomorphic functions over the entire boundary $*$ ∂D of a bounded domain $D \subset C^n$, $n \geqslant 1$, are obtained from formula (2):

A. If $n = 1$ and $\varphi(z_1, \bar{z}_1, \zeta_1) \equiv 1$, then formula (2) leads to the Cauchy formula.

B. Let $n = 1$, let the domain D be simply connected and such that the function $\omega(z_1)$, conformally mapping D onto the unit disk, has a derivative $\omega'(z_1)$ continuous in the closed domain $\overline{D}$. If we put

$$\varphi = \frac{\omega'(\zeta_1)(\zeta_1 - z_1)}{\omega(\zeta_1) - \omega(z_1)},$$

then formula (2) will give an integral representation that can be obtained from the Cauchy formula by means of the conformal mapping effected by the function $\omega(z_1)$. This same integral representation is easily transformed into an integral representation with the Szegő kernel (see, for example, (^{2})).

C. If $**$ $\varphi(z, \bar{z}, \zeta) \equiv 1$, the domain D is convex, and its boundary ∂D is regular (i.e. is a manifold of class C^∞), then formula (2) is another form of writing the Cauchy–Fantappiè integral representation indicated by Leray (^{3}); see also the papers of Norguet (^{4}). Thus, the integral representation (2) for $\varphi(z, \bar{z}, \zeta) \equiv 1$ may be regarded as an extension of the Cauchy–Fantappiè formula to the case where convexity of the domain D is not required, and the requirement of regularity of the boundary ∂D is replaced by a smoothness requirement ($***$).

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* For $n = 1$ the function φ is assumed to be defined on the set $\{(z, \xi) : z \in D,\ \xi \in \bar{D}\}$.

We also note that our proof of formula (2) does not require (in contrast to the proof of the Cauchy—Fantappiè formula given in Leray' s book (^{3})) the use of methods of algebraic topology (see (^{3}), pp. 93-99), but relies only on Stokes' formula (see, for example, (^{5}), § 6).

D. The Martinelli—Bochner integral representation (see (^{6}), § 21) is obtained from representation (2) for $\varphi(z, \bar{z}, \zeta) \equiv 1$ and

$$\Phi(z, \bar{z}, \zeta, \bar{\zeta}) = |\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2 + \cdots + |\zeta_n - z_n|^2.$$

E. Let the domain $D = \{z : \Psi(z, \bar{z}) < 0\}$ be convex, let the function Ψ be twice continuously differentiable, and let all first-order derivatives of Ψ not vanish simultaneously at points of the boundary ∂D. Then for $\varphi(z, \bar{z}, \zeta) \equiv 1$ and

$$\Phi = (\zeta_1 - z_1)\Psi'_{\zeta_1} + (\zeta_2 - z_2)\Psi'_{\zeta_2} + \cdots + (\zeta_n - z_n)\Psi'_{\zeta_n}$$

formula (2) leads to the integral representation obtained by us earlier (^{1}). In particular, for the hypersphere $\{z : |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 < 1\}$ one obtains the integral formula of Hua Lo-keng (^{7}).

F. If $n = 2$, and the doubly circular convex domain

$$D = \{z : |z_2| < \Psi(|z_1|)\}$$

is such that the function Ψ is twice continuously differentiable, then, putting $\varphi(z, \bar{z}, \zeta) \equiv 1$,

$$\Phi = 1 - \frac{z_1 \bar{\zeta}_1 \Psi'(|\zeta_1|)}{|\zeta_1| \left[|\zeta_1| \Psi'(|\zeta_1|) - \Psi(|\zeta_1|) \right]} + \frac{z_2 \bar{\zeta}_2 \Psi'(|\zeta_1|)}{|\zeta_2|^2 \left[|\zeta_1| \Psi'(|\zeta_1|) - \Psi(|\zeta_1|) \right]},$$

we arrive at Temlyakov' s integral representation, written in another form (see (^{8},^{9}); (^{6}), § 23).

$*$ Integral representations over manifolds of dimension smaller than the dimension of ∂D are proposed by the author for consideration in another note.

$**$ In sections C—D it is assumed that $n \geqslant 1$.

($***$) We point out that in paper (^{11}) formula (2) is derived from the Cauchy —Fantappiè formula (for $\varphi(z, \bar{z}, \zeta) \equiv 1$) for functions $f(z)$ holomorphic in a convex domain $D_1 \supset D$. From this result one could obtain formula (2) under the additional assumption that D is a Runge domain of the first kind.

Zh. Integral representations with the Cauchy kernels or with "extended Cauchy kernels" for bicircular domains ($n = 2$), found by the author earlier (see (^{9}); (^{6}), § 23), are also obtained from formula (2) for $\varphi(z, \bar{z}, \zeta) \equiv 1$ and a corresponding choice of the function Φ.

3. All integral formulas that can be obtained from the integral representations V—Zh by means of biholomorphic (pseudoconformal) mappings of the closed domain $\bar{D}$ are consequences* of formula (2) (here the functions φ are chosen holomorphic not only in ζ, but also in z).

4. We indicate some **new integral representations** that follow from the theorem stated above.

I. Let $\Psi(z, z')$ be an entire function of $2n$ complex variables $(z_1, z_2, \dots, z_n, z'_1, z'_2, \dots, z'_n)$, with the function $\Psi(z, \bar{z})$ real-valued and the domain $D = \{z : \Psi(z, \bar{z}) < 0\}$ bounded. We require that, for arbitrary z and z', the inequality

$$\Psi(z, \bar{z}) - \Psi(z, \bar{z}') - \Psi(z', \bar{z}) + \Psi(z', \bar{z}') \geq 0. \tag{3}$$

be satisfied.

Under these assumptions, as the function Φ in formula (2) one may take the function** $\Psi(z, \bar{\zeta})$, which will satisfy condition 1) by virtue of inequality (3) and condition 2) by virtue of Hefer' s theorem (see (^{10}), p. 137). The integral representation (2) obtained in this way is of interest in that the exterior form (1) is holomorphic in z, and in its denominator there stands the "equation" of the boundary ∂D of the domain D.

We give examples of domains satisfying condition (3).

a) Domains $D = \{z : \Psi(z, \bar{z}) < 0\}$, if the function Ψ has the following property: in each analytic plane $E^{(\alpha,\beta)} = \{z : z_1 = \alpha_1 t + \beta_1,\ z_2 = \alpha_2 t + \beta_2, \dots,\ z_n = \alpha_{nt} + \beta_n\}$ the equality $\Delta_t \Psi = C_{\alpha,\beta}$ holds (here Δ_t is the Laplace operator with respect to the complex parameter t, and $C_{\alpha,\beta}$ is a nonnegative constant depending only on (α, β)).

b) n-circular domains $D = \{z : |z_1|^{2m_1} + |z_2|^{2m_2} + \dots + |z_n|^{2m_n} - 1 < 0\}$, where $m_1, m_2, \dots, m_n$ are natural numbers.

K. Consider a domain $D = \{z : \Psi(z, \bar{z}) < 0\}$ that is strictly analytically convex in the sense of Hartogs (see (^{10}), p. 237), where the function Ψ is three times continuously differentiable. It can be shown that for each point $\zeta \in \partial D$ there exists an $\varepsilon > 0$ such that, whenever $|z_1 - \zeta_1|^2 + |z_2 - \zeta_2|^2 + \dots + |z_n - \zeta_n|^2 < \varepsilon,\ z \in D$, the inequality

$$\sum_{i=1}^{n}\frac{\partial\Psi(\zeta,\bar\zeta)}{\partial\zeta_i}(z_i-\zeta_i)+\sum_{i=1}^{n}\frac{\partial\Psi}{\partial\bar\zeta_i}(\bar z_i-\bar\zeta_i)+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\Psi}{\partial\zeta_j\partial\zeta_i}(z_i-\zeta_i)(z_j-\zeta_j)+$$

$$+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\Psi}{\partial\zeta_j\partial\bar\zeta_i}(z_i-\zeta_i)(\bar z_j-\bar\zeta_j)<0. \tag{4}$$

Therefore it is natural to call a domain D "strictly analytically convex as a whole" if inequality (4) holds for all $z \in D$, $\zeta \in \partial D$. For such domains the integral representation (2) is valid, where the function

$$\Phi=\sum_{i=1}^{n}\frac{\partial\Psi(\zeta,\bar\zeta)}{\partial\zeta_i}(z_i-\zeta_i)+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\Psi(\zeta,\bar\zeta)}{\partial\zeta_j\partial\zeta_i}(z_i-\zeta_i)(z_j-\zeta_j)$$

is a polynomial of the second degree with respect to z.

* See below, property 2.

** The function φ in sections I–L may be arbitrary (under conditions 3)–4)).

L. If the domain $D=\{z:\Psi(z,\bar z)<0\}$ is strictly analytically convex in the sense of Hartogs (see (^{10}), p. 237), then for constructing the integral representation (2) one may use the following function:

$$\Phi(z,\bar z,\zeta,\bar\zeta)=\sum_{i,j=1}^{n}\frac{\partial^2\Psi(\zeta,\bar\zeta)}{\partial\zeta_i\partial\bar\zeta_j}(z_i-\zeta_i)(\bar z_j-\bar\zeta_j).$$

4. Exterior differential forms μ of the form (1) have the following properties:

Property 1. If μ_1, μ_2 are exterior differential forms of the form (1), and $h_1(z,\bar z,\zeta)$, $h_2(z,\bar z,\zeta)$ are functions continuous on the set $\{(z,\zeta): z\in D,\ \zeta\in\partial D\}$, holomorphic in ζ, with $h_1+h_2\neq 0$, then the exterior differential form

$$\frac{h_1\mu_1+h_2\mu_2}{h_1+h_2}$$

can also be reduced to the form (1).

Hence, in particular, it follows (if the functions h_1 and h_2 are taken to be constants) that exterior differential forms of the form (1) form an affine linear manifold in the vector space (over the field of complex numbers) of all exterior differential forms on ∂D with continuous coefficients.

Property 2. Every exterior differential form μ of the form (1), under a biholomorphic mapping of the closed domain $\bar D$, is transformed into a form of the same kind.

5. Condition 2) imposed on the function $\Phi(z, \bar{z}, \zeta, \bar{\zeta})$ in item 1 can be replaced by the more general condition:

2′) $\Phi(z, \bar{z}, \zeta, \bar{\zeta})$ can be represented in the form $\Phi = [Q_1(\zeta) - Q_1(z)]P_1 + [Q_2(\zeta) - Q_2(z)]P_2 + \dots + [Q_n(\zeta) - Q_n(z)]P_n$, where the functions Q_i are holomorphic in the closed domain $\bar{D}$, and the functions P_i have continuous derivatives with respect to $(\zeta, \bar{\zeta})$, $i = 1, 2, \dots, n$.

If, on the basis of Hefer's theorem (see (^{10}), p. 137), one represents $Q_i(\zeta) - Q_i(z)$ in the form

$$Q_i(\zeta) - Q_i(z) = (\zeta_1 - z_1)q_{i1} + (\zeta_2 - z_2)q_{i2} + \dots + (\zeta_n - z_n)q_{in}, \quad i = 1, 2, \dots, n,$$

where the functions $q_{ij}(\zeta, z)$ are holomorphic for $\zeta \in \bar{D},\ z \in \bar{D}$, then from the theorem of item 1 one obtains

Corollary. If the function $f(z)$ is holomorphic in the domain D and continuous in the closed domain $\bar{D}$, then for points $z \in D$ the formula

$$f(z) = \int_{\partial D} f(\zeta)\, \Omega \mu$$

holds, where the exterior differential form μ is defined by equality (1),

$$\Omega = \begin{vmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ \vdots & \vdots & & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{vmatrix}.$$

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Note: Figure translations are in progress. See original paper for figures.

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