# E. V. MAIKOV

1964

SovietRxiv

**Abstract**

**Full Text**



# E. V. MAIKOV



# $\tau$-CONTINUITY AND $\tau$-DIFFERENTIABILITY OF A FUNCTIONAL

*(Presented by Academician P. S. Aleksandrov, 25 XI 1963)*

For functionals defined on the set of piecewise-continuous functions $\xi(\tau)$, $\tau \in D = [0;1]$, this note introduces and considers notions which we call $\tau$-continuity and $\tau$-differentiability. They are connected with a perturbation of the independent variable $\xi(\tau)$, localized in a small neighborhood of an arbitrarily fixed $\tau$. These notions were used by the author (see (1)) in generalizing the concept of a measure in a functional space. One may expect that they will also be useful in other questions (for example, in the theory of optimal control, etc.).

**The space $\Xi$.** Let $\xi(\tau)$, $\tau \in D$, be a real piecewise-continuous function such that $\xi(\tau+0) = \xi(\tau)$, $\xi(1-0) = \xi(1)$. The set of points $\tau \in D$ at which $\xi(\tau)$ is discontinuous will be denoted by $C_\xi$. Let $\Xi = \{\xi(\tau)\}$. Elements of $\Xi$ will be denoted by $\xi$ (when necessary, with an upper index).

Take $\xi^0, \xi^1 \in \Xi$. Denote (for $0 < \theta < 1$)

$$\xi^\theta = \xi^\theta(\tau) = \begin{cases} \xi^1(\tau), & \text{for } 0 \le \tau < \theta, \\ \xi^0(\tau), & \text{for } \theta \le \tau \le 1. \end{cases} \tag{1}$$

The set $[\xi^0;\xi^1] = \{\xi^\theta; 0 \le \theta \le 1\}$ will be called the $\tau$-segment joining the point $\xi^0$ with the point $\xi^1$. Note that $[\xi^0;\xi^1] = [\xi^1;\xi^0]$ if and only if $\xi^0 = \xi^1$.

**$\tau$-continuity.** Let an arbitrary complex-valued functional $f(\xi)$ be given on $\Xi$. Take some $\xi^0 \in \Xi$. Fix $t \in D$ and $x \in R = (-\infty;+\infty)$, and choose $\xi \in \Xi$ such that $\xi(t-0) = \xi(t) = x$. Denote

$$\xi^\eta = \begin{cases} \xi(\tau), & \text{for } \tau \in [t;t+\eta), \\ \xi^0(\tau), & \text{for the remaining } \tau \in D \end{cases} \tag{2}$$

(for $\eta < 0$ we take $[t;t+\eta) = \{\tau; t+\eta \le \tau < t\}$). If, independently of the choice of $\xi(\tau)$, the equality

$$\lim_{\eta \to +0} f(\xi^\eta) = f(\xi^0)$$

holds, then $f(\xi)$ is called $\tau$-continuous from the right at the point $\xi^0$ with respect to the pair $(t, x)$. $\tau$-continuity from the left is defined analogously. If, at the point $\xi^0$, the functional $f(\xi)$ is $\tau$-continuous from the left and from the right with respect to all pairs $(t, x)$, then it is called $\tau$-continuous at the point $\xi^0$. If $f(\xi)$ is $\tau$-continuous at every point, we shall simply say that it is $\tau$-continuous or, more briefly, $f(\xi) \in T_0$.

A functional $\tau$-continuous at the point $\xi^0$ from the right with respect to the pair $(t, \xi^0(t))$ and from the left with respect to the pair $(t, \xi^0(t-0))$ will be called elementarily continuous for $\xi = \xi^0$ with respect to $\tau = t$.

**Example 1.** The functional

$$f(\xi) = \int_0^1 \xi(t)\, dt$$

is $\tau$-continuous.

**Example 2.** The functional $f(\xi) = l\{\tau; \xi(\tau) > 0\}$, where $l$ is Lebesgue measure on the interval $D$, is $\tau$-continuous.

**Example 3.** The functional $f(\xi) = \sup_\tau \xi(\tau)$ is not $\tau$-continuous at any point $\xi \in \Xi$ (but it is elementarily continuous).

## Properties of $\tau$-continuous functionals.

1. The functional $f(\xi)$ is $\tau$-continuous if and only if it is continuous on every $\tau$-segment, i.e., if for any $\xi^0, \xi^1$ the function of one variable

$$F(\theta) = F_{\xi^0 \xi^1}(\theta) = f(\xi^\theta), \tag{3}$$

is continuous, where $\xi^\theta$ is the same as in (1).

2. Whatever the numbers $z_j$ and the (distinct) points $\xi^j \in \Xi$ $(j = 1, \dots, n)$, there exists an $f(\xi)$ such that $f(\xi^j) = z_j$.
3. If the functionals $f_1, f_2 \in T_0$, then $\lambda_1 f_1 + \lambda_2 f_2$, $f_1 \cdot f_2 \in T_0$, and, for $f_2 \neq 0$, $f_1 : f_2 \in T_0$.
4. If $f(\xi) \in T_0$, and $w(z)$ is a continuous function of a complex variable, then $w(f(\xi)) \in T_0$.
5. If $f_1, f_2, \dots \in T_0$ and $f_n \Rightarrow f$, then $f \in T_0$.

## Relation to other definitions of continuity of a functional.

Denote by $C_0$ the totality of functionals continuous in the sense of the uniform topology on $\Xi$. As Examples 1-3 show, each of the sets $C_0 \cap T_0$, $\overline{C}_0 \cap T_0$, $C_0 \cap \overline{T}_0$ is nonempty. Obviously, $\overline{C}_0 \cap \overline{T}_0$ is also nonempty. If the topology $L_p$ is introduced in $\Xi$, then from the continuity of $f(\xi)$ in this topology its $\tau$-continuity follows. The converse is false (Example 2).

## $\tau$-Differentiability.

Let $\xi^0$ be a fixed point of $\Xi$; $\xi^\eta$ is the same as in (2). If there exists, independent of the choice of $\xi$,

$$\lim_{\eta \to +0} \frac{f(\xi^\eta) - f(\xi^0)}{\eta} = f_+^{(1)}(\xi^0, t, x), \tag{4}$$

then $f(\xi)$ is called **$\tau$-differentiable from the right at the point $\xi^0$ with respect to the pair** $(t, x)$, and this limit is the **right $\tau$-derivative** of $f(\xi)$ at the point $\xi^0$ with respect to $(t, x)$. Similarly, the **left $\tau$-derivative** is the limit (if it exists)

$$\lim_{\eta \to -0} \frac{f(\xi^\eta) - f(\xi^0)}{-\eta} = f_-^{(1)}(\xi^0, t, x). \tag{5}$$

If the limits (4) and (5) exist for $(t, x) \in (D \times R)$, coincide for $t \in C_{\xi^0}$, and for the remaining $t$ (their number is finite for each $\xi^0$) the "jump condition"

$$f_+^{(1)}(\xi^0, t, \xi^0(t-0)) + f_-^{(1)}(\xi^0, t, \xi^0(t)) = 0,$$

is fulfilled, then $f(\xi)$ is called **$\tau$-differentiable at the point** $\xi^0$, and the function of two variables defined on $C_{\xi^0} \times R$,

$$f^{(1)}(\xi^0, t, x) = f_+^{(1)}(\xi^0, t, x) = f_-^{(1)}(\xi^0, t, x)$$

is called the **$\tau$-derivative of the functional** $f(\xi)$ **at the point** $\xi^0$ and is denoted by $\delta f / \delta \tau$, or, more briefly, $f^{(1)}$. Everywhere below we shall assume that this function is extended in such a way that for $t \notin C_{\xi^0}$ it is equal to $f_+^{(1)}(\xi^0, t, x)$. If $f(\xi)$ has a $\tau$-derivative for all $\xi^0 \in \Xi$, we shall simply say that it is $\tau$-differentiable, or, more briefly, $f(\xi) \in T_1$.

**Example 4.** Let

$$f(\xi) = \int_0^1 \varphi(t, \xi(t))\, dt,$$

where $\varphi(t, x)$ is a continuous function on $D \times R$. Then $f(\xi) \in T_1$ and

$$f^{(1)}(\xi, t, x) = \varphi(t, x) - \varphi(t, \xi(t)).$$

## Properties of $\tau$-differentiable functionals.

1. $T_1 \subset T_0$.
2. If $f(\xi) \in T_1$, then for any $\xi^0, \xi^1$ the function $F(\theta)$ (see (3)) is continuous and, for $\theta \in C_{\xi^0} \cap C_{\xi^1}$, differentiable, so that

$$F'(\theta) = f^{(1)}(\xi^\theta, \theta, \xi^1(\theta)).$$

3. If $f_1, f_2 \in T_1$, then $\lambda_1 f_1 + \lambda_2 f_2,\ f_1 \cdot f_2 \in T_1$, and, for $f_2 \neq 0$, $f_1 : f_2 \in T_1$, with

$$(\lambda_1 f_1 + \lambda_2 f_2)^{(1)} = \lambda_1 f_1^{(1)} + \lambda_2 f_2^{(1)}, \qquad (f_1 \cdot f_2)^{(1)} = f_1^{(1)} \cdot f_2 + f_1 \cdot f_2^{(1)},$$

$$\left(\frac{f_1}{f_2}\right)^{(1)} = \frac{f_1^{(1)} \cdot f_2 - f_1 \cdot f_2^{(1)}}{f_2^2}$$

(analogous equalities are also satisfied by the left $\tau$-derivatives).

4. If $f(\xi) \in T_1$, and $w(z)$ is a differentiable function of a complex variable, then $g(\xi) = w(f(\xi)) \in T_1$ and $g^{(1)} = w'(f) \cdot f^{(1)}$.
5. If $f(\xi) \in T_1$ and $f^{(1)}(\xi, t, x) \equiv 0$, then $f(\xi) = \text{const}$.
6. For any $f(\xi) \in T_1$, for all $\xi, t$, the equality

$$f^{(1)}(\xi, t, \xi(t)) = 0$$

holds.

7. If $f(\xi) \in T_1$ attains a maximum (minimum) at $\xi = \xi^0$, then $f^{(1)}(\xi^0, t, x) \leq 0$ $(\geq 0)$ for all $t, x$.

From properties 5 and 6 there follows the curious consequence: if $f^{(1)} \equiv \text{const}$ (or even only does not depend on $\xi$, or does not depend on $x$), then $f^{(1)} \equiv 0$, i.e. $f(\xi) \equiv \text{const}$.

**Finite-dimensional analogue of the $\tau$-derivative.** Let $R^n = \{\xi = (\xi_1, \dots, \xi_n)\}$ be an $n$-dimensional space and $f(\xi) = f(\xi_1, \dots, \xi_n)$. The analogue of the concept of the $\tau$-derivative for a function $f(\xi)$ is the expression

$$f^{(1)}(\xi, k, x) = f(\xi_1, \dots, \xi_{k-1}, x, \xi_{k+1}, \dots, \xi_n) - f(\xi_1, \dots, \xi_{k-1}, \xi_k, \xi_{k+1}, \dots, \xi_n).$$

Thus, the finite-dimensional analogue of the $\tau$-derivative is not (in contrast, for example, to the variational derivative) an infinitesimal concept. Every function of a finite number of variables is "$\tau$-differentiable," and a passage to the limit appears only in the case of an infinite-dimensional space.

**Connection with the variational derivative.** Let $f(\xi) \in T_1$ and at the same time have the variational derivative $\delta f / \delta \xi$ at the point $\xi^0$. Suppose, moreover, that $t \in C_{\xi^0}$ and the function $f^{(1)}(\xi^0, t, x)$ is differentiable with respect to $x$ at $x = \xi^0(t)$. Then

$$\frac{\delta f}{\delta \xi} = \frac{\partial}{\partial x}\left(\frac{\delta f}{\delta t}\right)\Big|_{x=\xi^0(t)}.$$

**Continuous $\tau$-derivative.** In this note we shall call a $\tau$-derivative $f^{(1)}$ continuous if, for each $\xi$, it is continuous jointly in the variables $(t, x)$ for $t \in C_\xi$, and if, moreover, on each $\tau$-segment the expression $f^{(1)}(\xi^0_\theta, \theta, \xi^1(\theta))$ is continuous for $\theta \in C_{\xi^0} \cap C_{\xi^1}$ and has finite left and right limits for the remaining $\theta$.

**$\tau$-Integral.** Can one reconstruct the functional $f$ from a given $\tau$-derivative $f^{(1)}$? This question is answered by

**Theorem 1.** *If $f(\xi) \in T_1$ and, for any fixed $\xi^0, \xi^1$, the expression $f^{(1)}(\xi^0_\theta, \theta, \xi^1(\theta))$ is bounded, then*

$$f(\xi^1) - f(\xi^0) = \int_0^1 f^{(1)}(\xi^0_\theta, \theta, \xi^1(\theta))\, d\theta = \int_{\xi^0}^{\xi^1} \frac{\delta f}{\delta \tau}\, d\tau, \tag{6}$$

*i.e. the $\tau$-derivative determines the functional up to an additive constant.*

On the right-hand side of (6) we have introduced a new notation (the $\tau$-integral), clear from the formula itself.

It is easy to see that (6), in particular, is valid for continuously differentiable functionals.

**$\tau$-Derivatives of higher orders.** The first $\tau$-derivative $f^{(1)}(\xi, t_1, x_1)$ of a $\tau$-differentiable functional, for arbitrary fixed $(t_1, x_1)$, is a functional of $\xi$. Let us take one more pair of numbers $(t_2, x_2)$ and compute the $\tau$-derivative of $f^{(1)}$ with respect to this pair.

We shall say that $f(\xi)$ is twice $\tau$-differentiable if $f^{(1)}$ is $\tau$-differentiable for all $(t_2, x_2)$, except, possibly, $t_2 = t_1$, and is elementarily continuous for every $\xi$ with respect to $\tau = t_1$. Further, if the $(n-1)$-st $\tau$-derivative $f^{(n-1)}(\xi, t_1, x_1, \dots, t_{n-1}, x_{n-1})$ of the functional $f(\xi)$ is elementarily continuous with respect to $t_1, \dots, t_{n-1}$ and is $\tau$-differentiable for $t_n \neq t_1, \dots, t_{n-1}$, then $f(\xi)$ is called $n$ times $\tau$-differentiable, and

$$\frac{\delta}{\delta \tau}\left(\frac{\delta^{n-1} f}{\delta \tau^{n-1}}\right) = \frac{\delta^n f}{\delta \tau^n} = f^{(n)}(\xi, t_1, x_1, \dots, t_n, x_n)$$

is called its $n$-th $\tau$-derivative.

The continuity of the $n$-th $\tau$-derivative is defined analogously to the continuity of the first $\tau$-derivative.

**Theorem 2.** If $f(\xi)$ is continuously $\tau$-differentiable and $f^{(2)} \equiv 0$, then $f(\xi)$ can be represented in the form

$$f(\xi) = \int_0^1 \varphi(t, \xi(t))\, dt,$$

where $\varphi(t, x)$ is some function of two variables, continuous on $D \times R$.

**Proof.** Find $f^{(1)}(\xi, t_1, x_1)$, regarding $(t_1, x_1)$ as fixed. Take two points $\xi^0$ and $\xi^1$ such that $\xi^0(t_1 \pm 0) = \xi^1(t_1 \pm 0)$. Consider the function $F(\theta) = f^{(1)}(\xi^\theta, t_1, x_1)$. Using the definition of the second $\tau$-derivative, we find that for all $\theta$ this function is continuous and is differentiable everywhere except, possibly, at a finite number of points, and moreover $F'(\theta) = 0$. Consequently, $F(\theta) = \text{const}$, i.e.

$$f^{(1)}(\xi^0, t_1, x_1) = f^{(1)}(\xi^1, t_1, x_1).$$

Hence

$$f^{(1)}(\xi, t_1, x_1) = \psi(t_1, x_1, \xi(t_1 - 0), \xi(t_1)),$$

where $\psi$ is a function of real variables, defined on $D \times R^3$. Now using Theorem 1 and putting $\xi^0 = 0$ in (6), we obtain

$$f(\xi^1) = \int_0^1 \varphi(t, \xi(t))\, dt,$$

where $\varphi(t, x) = f(0) + \psi(t, x, x, 0)$. We shall not dwell on the proof of the continuity of $\varphi$.

Similarly one proves

**Theorem 3.** If $f(\xi)$ is $n$ times continuously $\tau$-differentiable and $f^{(n+1)} \equiv 0$, then

$$f(\xi) = \int_0^1 \cdots \int_0^1 \varphi(t_1, \xi_1, \dots, t_n, \xi_n)\, dt_1 \cdots dt_n, \tag{7}$$

where $\xi_i = \xi(t_i)$, and $\varphi(t_1, x_1, \dots, t_n, x_n)$ is some continuous function defined on $(D \times R)^n$.

**Theorem 4.** A functional of the form (7), for any continuous function $\varphi(t_1, x_1, \dots, t_n, x_n)$, has continuous $\tau$-derivatives of all orders, and moreover $f^{(n+1)} \equiv 0$.

The last theorems show that functionals of the form (7) possess, with respect to $\tau$-differentiation, the properties of polynomials. We call them $\tau$-polynomials. Some of their properties have been used by us in constructing the notion of $\tau$-analytic functionals.

I am pleased to express my gratitude to Prof. S. V. Fomin for numerous discussions of questions related to this work.

Moscow State University
named after M. V. Lomonosov

Received
18 IX 1963

## REFERENCES

1. E. V. Maikov, UMN, **18**, no. 3, 243 (1963).

*Note: Figure translations are in progress. See original paper for figures.*