



Soviet-era science, translated into English

V. E. Lyantse

1964

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196401.10471>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

V. E. Lyantse

ON A NON-SELF-ADJOINT DIFFERENTIAL OPERATOR OF THE SECOND ORDER ON THE HALF-AXIS

(Presented by Academician L. S. Pontryagin on 21 X 1963)

Let $l[y] = -y'' + p(x)y$, $\sigma_1(x) = \int_x^\infty u |p(u)| du$. In the monograph of Z. S. Agranovich and V. A. Marchenko ⁽¹⁾ it is proved that if for every $x \geq 0$ $\sigma_1(x) < \infty$, then the equation $l[y] = s^2 y$ has a solution $y_1(x, s)$, which exists for $x \geq 0$, $\text{Im } s \geq 0$, and admits the representation

$$y_1(x, s) = e^{ixs} + \int_x^\infty K(x, t)e^{its} dt. \quad (1)$$

with kernel $K(x, t)$, satisfying the inequality

$$\int_x^\infty |K(x, t)| dt \leq e^{\sigma_1(x)} \sigma_1(x) \quad (2)$$

(see also ⁽²⁾). From estimate (2) it follows that for every x for which $\sigma_1(x) < \infty$, function (1) is continuous in s for $\text{Im } s \geq 0$ and holomorphic in s for $\text{Im } s > 0$. Some questions of spectral theory become simpler if function (1) can be analytically continued beyond the axis $\text{Im } s = 0$. Thus, in the work of M. A. Naimark ⁽³⁾, and then in the author's works ⁽⁴⁻⁶⁾, the fact is systematically used that if the function $p(x) \exp(\varepsilon x)$ is summable on the half-axis $x > 0$, then for every $x \geq 0$ function (1) is holomorphic in s for $\text{Im } s > -\varepsilon/2$. V. B. Lidskii drew the author's attention to the fact that the enlargement of the domain of analyticity of function (1) may occur not only due to a sufficiently rapid decrease of the potential $p(x)$, but also due to its regularity near the infinitely remote point. This remark served as the stimulus for proving the following proposition.

Let X denote the domain in the complex x -plane defined by the relation

$$X = \{x : x = \rho e^{i\varphi}, \rho > \rho_0, |\varphi| < \varphi_0\}, \quad (3)$$

where ρ_0 and φ are certain positive numbers. Suppose that the potential $p(x)$ is a (complex-valued) function defined in the domain $(0, \infty) \cup X$, summable on the half-axis $(0, \infty)$, holomorphic and satisfying the inequality

$$|p(x)| < \frac{A}{|x|^\alpha}, \quad \alpha > 2, \quad (4)$$

in the domain X . Then for all $x \geq 0$ $\sigma_1(x) < \infty$, and consequently the differential equation $l[y] = s^2y$ has a solution $y_1(x, s)$, which for $x \geq 0$, $\text{Im } s \geq 0$ admits representation (1).

Theorem 1. Under the assumptions indicated above, the kernel $K(x, t)$ can be continued from the real domain $W_0 = \{(x, t) : 0 \leq x \leq t < \infty\}$ to a function holomorphic ($\overline{\text{in } x, t}$) in the complex domain W_1 , consisting of such pairs (x, t) that the ray $x\overline{t\infty}$ with initial point at x , passing through the point t ,

t , belongs to the domain X (see relation (3)) and forms with the ray $\overline{Ox\infty}$ an angle not greater than a straight angle. The analytic continuation of the kernel $K(x, t)$ satisfies the inequality

$$|K(x, t)| \leq \frac{B}{|x + t|^{\alpha-1}}, \quad (x, t) \in W_1, \quad (5)$$

where B is some constant. Moreover, for every $x \geq 0$ the function (1) can be continued from the half-plane $\text{Im } s \geq 0$ to a function holomorphic in the domain (see relation (3))

$$S_0 = \{s : s = \sigma e^{i\tau}, \sigma > 0, -\varphi_0 < \tau < \pi + \varphi_0\}, \quad (6)$$

while remaining a solution of the differential equation $l[y] = s^2y$. The analytic continuation of the function (1) into the half-plane $\text{Im } se^{i\psi} > 0$, $|\psi| < \varphi_0$, can be represented in the form

$$y_1(x, s) = e^{ixs} + \int_{\overline{xx_1\infty_\psi}} K(x, t)e^{its} dt, \quad (7)$$

where x_1 is some real point of the domain X , and $\overline{xx_1\infty_\psi}$ is a broken line composed of the segment with endpoints x and x_1 and the ray with initial point at x_1 , making an angle ψ with the axis Ox . For every s for which $\text{Im } se^{i\psi} > 0$, $|\psi| < \varphi_0$, the function (1) admits analytic continuation in x to the domain X , which is also represented by formula (7).

To prove the present theorem we apply the well-known method (see, for example, (7)) of shifting into the complex domain the contour of integration in the integral equation for determining the kernel $K(x, t)$.

Let us now consider the differential operator L_θ , defined in the Hilbert space $L^2(0, \infty)$ by the differential expression $l[y]$ and the boundary condition $y'(0) - \theta y(0) = 0$, where θ is some complex number. As is known (see (3)), the eigenvalues of the operator L_θ are determined by the relations $\lambda = s^2$, $\text{Im } s > 0$,

$A(s) = 0$, where $A(s) = y'_{1x}(0, s) - \theta y_1(0, s)$. The so-called spectral singularities of the operator L_θ (see (4)) are determined by the relations $\lambda = \sigma^2$, $\text{Im } \sigma = 0$, $A(\sigma) = 0$.

Theorem 2. If the potential $p(x)$ satisfies the conditions of Theorem 1, then the operator L_θ has at most a countable set of eigenvalues and spectral singularities, and the only limit point for this set can be the point zero. If, moreover, in estimate (4) $\alpha > 3$, then the set of eigenvalues and spectral singularities of the operator L_θ is finite.

The proof is based on the fact that, by Theorem 1, the function $A(s)$ is holomorphic for $s \in S_0$ (see relation (6)) and $A(s) = is[1 + o(1)]$ as $|s| \rightarrow \infty$. If in estimate (4) $\alpha > 3$, then it is not hard to show that the function $A(s)$ also has a derivative at the vertex $s = 0$ of the angle S_0 . On the other hand, the equalities $A(0) = 0$, $A'(0) = 0$ are incompatible with the fact that the Wronskian of the functions $y_1(x, s)$, $y_1(x, -s)$ is equal to $-2is$. Therefore the point 0 cannot be a limit point for the set of roots of the equation $A(s) = 0$.

If the potential $p(x)$ satisfies the conditions of Theorem 1 with $\alpha > 3$, then for the operator L_θ all the assertions contained in Theorems 1-4, formulated in our article (4), remain valid. In particular, the operator L_θ with spectral singularities is a generalized spectral operator* in the sense indicated in (8) (see also (9)).

Denote by $\omega(x, \lambda)$ the solution of the equation $l[y] = \lambda y$ satisfying the initial conditions $\omega(0, \lambda) = 1$, $\omega'_x(0, \lambda) = \theta$. By the l_θ -transform

* But it is not spectral in the usual sense.

The Fourier transform of a function $f \in L^2(0, \infty)$ is the function

$$\omega(f, \lambda) = \int_0^\infty f(x) \omega(x, \lambda) dx$$

of the variable $\lambda > 0$ and the collection of numbers

$$\omega^{(j)}(f, \lambda_k) = \int_0^\infty f(x) \omega^{(j)}(x, \lambda_k) dx,$$

$j = 0, \dots, m_k - 1$, $k = 1, \dots, r$; here $\lambda_1, \dots, \lambda_r$ are the eigenvalues of the operator L_θ ; m_1, \dots, m_r are their multiplicities, and $\omega^{(j)}$ denotes the derivative of order j of the function ω with respect to the variable λ .

M. A. Naimark, in the paper ⁽³⁾, constructed inversion formulas for the l_θ -Fourier transform under the assumption that the function $(1 + x^2)p(x)$ is summable on the real axis and that the operator L_θ has no spectral singularities. B. Ya. Levin in the paper ⁽²⁾ and V. A. Marchenko in the paper ⁽¹⁰⁾ showed that M. A. Naimark's formulas remain valid under weaker restrictions.

Inversion formulas for the case when the operator L_θ has spectral singularities and the function $p(x)\exp(\varepsilon x)$ is summable on the half-axis $x > 0$ for some $\varepsilon > 0$ were constructed in our paper ⁽⁵⁾. These formulas contain operators acting from a certain space of generalized functions into the space $L^2(0, \infty)$. Under the assumptions adopted in the present paper, the use of generalized functions proves difficult, since the kernel $K(x, t)$ decreases insufficiently rapidly at infinity. Therefore here we indicate another form of writing the inversion formulas, suitable both under the conditions of the present paper and under the assumptions adopted in the paper ⁽⁵⁾.

Put

$$\omega^*(x, \lambda) = \omega(x, \lambda) - \sum_{k=1}^{\rho} \sum_{j=0}^{\mu_k-1} B_{kj}(\lambda) \omega^{(j)}(x, \tilde{\lambda}_k), \quad (8)$$

where $\tilde{\lambda}_1, \dots, \tilde{\lambda}_\rho$ are the spectral singularities of the operator L_θ ; μ_1, \dots, μ_ρ are their multiplicities, and $B_{kj}(\lambda)$ are arbitrary functions, square-integrable on the half-axis $\lambda > 0$, differentiable $\mu_{k'} - 1$ times at the point $\tilde{\lambda}_{k'}$, $k' = 1, \dots, \rho$, and satisfying the relations

$$\left\{ \left(\frac{d}{d\lambda} \right)^{j'} B_{kj}(\lambda) \right\}_{\lambda=\tilde{\lambda}_{k'}} = \begin{cases} 1, & \text{if } j' = j, k' = k, \\ 0, & \text{in the remaining cases,} \end{cases} \quad (9)$$

$$j = 0, \dots, \mu_k - 1; \quad j' = 0, \dots, \mu_{k'} - 1; \quad k, k' = 1, \dots, \rho.$$

Theorem 3. For every function $f \in L^2(0, \infty)$ the formula* is valid

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\omega(f, \lambda) \omega^*(x, \lambda) \sqrt{\lambda}}{A(\sqrt{\lambda})A(-\sqrt{\lambda})} d\lambda + \sum_{k=1}^{\rho} \sum_{j=0}^{\mu_k-1} C_{kj} \omega^{(j)}(x, \tilde{\lambda}_k) + \sum_{k=1}^r \left\{ \left(\frac{d}{d\lambda} \right)^{m_k-1} M_k(\lambda) \omega(f, \lambda) \omega(x, \lambda) \right\}_{\lambda=\lambda_k}. \quad (10)$$

The integral in formula (10) exists in the sense of the norm

$$\| \cdot \|_\alpha = \left\{ \int_0^\infty \left| \frac{\cdot(x)}{(1+x)^\alpha} \right|^2 dx \right\}^{1/2}, \quad (11)$$

* The last term on the right-hand side of this formula is written in abbreviated form. To obtain the expanded notation one should carry out the differentiation with respect to λ , assuming that the function $\omega(f, \lambda)$ is differentiable, and then replace the (generally speaking, nonexistent) derivatives

$$\left\{ \left(\frac{d}{d\lambda} \right)^j \omega(f, \lambda) \right\}_{\lambda=\lambda_k}$$

by the numbers

$$\omega^{(j)}(f, \lambda_k) = \int_0^\infty f(x) \omega^{(j)}(x, \lambda_k) dx.$$

for any $a > \max(\mu_1, \dots, \mu_\rho)$, and the numbers C_{kj} are uniquely determined by the requirement that the sum of the first two terms on the right-hand side of formula (10) be a function square-integrable on the semiaxis $x > 0$; moreover, $M_k(\lambda) = (\lambda - \lambda_k)^{m_k} y_1(0, \sqrt{\lambda}) / (m_k - 1)! A(\sqrt{\lambda})$.

We note that if the functions $B_{kj}(\lambda)$ are chosen so that

$$B_{kj}(\lambda) = \begin{cases} \frac{(\lambda - \tilde{\lambda}_k)^j}{j!}, & \text{for } |\lambda - \tilde{\lambda}_k| < \delta, \\ 0, & \text{for the remaining values,} \end{cases} \quad (12)$$

where $\delta > 0$ is an arbitrary sufficiently small number, then the integral on the right-hand side of formula (10) reduces to the following simple form:

$$\int_{\Lambda_\delta} \omega(f, \lambda) \omega(x, \lambda) \frac{\sqrt{\lambda} d\lambda}{A(\sqrt{\lambda})A(-\sqrt{\lambda})} + \sum_{k=1}^\rho \int_{\tilde{\lambda}_k-\delta}^{\tilde{\lambda}_k+\delta} \omega(f, \lambda) \left[\omega(x, \lambda) - \sum_{j=0}^{\mu_k-1} \frac{(\lambda - \tilde{\lambda}_k)^j}{j!} \omega^{(j)}(x, \tilde{\lambda}_k) \right] \frac{\sqrt{\lambda} d\lambda}{A(\sqrt{\lambda})A(-\sqrt{\lambda})}. \quad (13)$$

Here Λ_δ is the complement of the (nonintersecting) δ -neighborhoods of the points $\tilde{\lambda}_1, \dots, \tilde{\lambda}_\rho$ on the semiaxis $\lambda > 0$.

The proof of formulas (10) is based on the analytic expression for the (generalized) spectral function of a non-self-adjoint operator obtained by V. A. Marchenko¹⁰.

Starting from the inversion formula (10), one can construct an operational calculus of functions of the operator L_0 , which makes it possible to study boundary-value problems for the corresponding partial differential equations in the quarter-plane (cf. with⁶).

Lviv Polytechnic Institute

Received
17 X 1963

CITED LITERATURE

- ¹ Z. S. Agranovich, V. A. Marchenko, *The Inverse Problem of Scattering Theory*, Kharkov, 1960.
- ² B. Ya. Levin, DAN, 106, No. 3 (1956).
- ³ M. A. Naimark, *Trudy Moskov. Matem. obshch.*, 3, 181 (1954).
- ⁴ V. E. Lyantse, DAN, 149, No. 2 (1963).
- ⁵ V. E. Lyantse, DAN, 150, No. 5 (1963).
- ⁶ V. E. Lyantse, DAN, 152, No. 4 (1963).
- ⁷ F. Tricomi, *Differential Equations*, Moscow, 1962.
- ⁸ V. E. Lyantse, DAN, 142, No. 2 (1963).
- ⁹ V. E. Lyantse, *Matem. sborn.*, 61 (103), 1 (1963).
- ¹⁰ V. A. Marchenko, *Matem. sborn.*, 52 (94), 2 (1960).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.