

# ON THE STARLIKENESS OF ARCS OF LEVEL LINES UNDER UNIVALENT CONFORMAL MAPPINGS

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**Abstract**

**Full Text**

**MATHEMATICS**

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**ON THE STARLIKENESS OF ARCS OF  
LEVEL LINES UNDER UNIVALENT CON-  
FORMAL MAPPINGS**

*(Presented by Academician M. A. Lavrent'ev, 7 XII 1963)*

For the class  $S$  of holomorphic functions univalent in the disk  $|z| < 1$ ,

$$w = f(z), \quad f(0) = 0, \quad f'(0) = 1,$$

the inequalities are well known

$$\underline{R} = \frac{r}{(1+r)^2} \leq |f(re^{i\varphi})| \leq \frac{r}{(1-r)^2} = \bar{R},$$

which indicate the minimal annulus in which all level lines

$$L(f, r) = \{w \mid w = f(re^{i\varphi}), \quad -\pi < \varphi \leq \pi\}$$

of functions  $f(z) \in S$  are situated for fixed  $r$ ,  $0 < r < 1$ . I. E. Bazilevich and G. V. Koričičiĭ <sup>(1)</sup> singled out in this annulus the ring of starlikeness, proving the existence of an absolute constant  $d_s$  ( $0.1005 \leq d_s < 0.134$ ) such that every arc of the line  $L(f, r)$  lying in the annulus

$$d_s \bar{R} \leq |w| \leq \bar{R}$$

is starlike\* for every function  $f(z)$  of the class  $S$ , and such that in the enlarged annulus

$$(d_s - \varepsilon) \bar{R} \leq |w| \leq \bar{R}, \quad \varepsilon > 0,$$

the line  $L(f, r)$ , for some function  $f(z)$  from  $S$ , will have a non-starlike arc.

Along with the problem of determining the exact value of the constant  $d_s$ , I. E. Bazilevich and G. V. Koričičiĭ posed (see <sup>(2)</sup>) the following problem: for each  $r$ ,  $r_s = \text{th } \pi/4 < r < 1$ , find

$$a_s(r) = \inf a(r)$$

over all  $a(r) < 1$  having the property that any arc of the line  $L(f, r)$  belonging to the annulus

$$a(r) \bar{R} \leq |w| \leq \bar{R}$$

is starlike, whatever the function  $f(z) \in S$ .

In this article we give a formula determining  $a_s(r)$  for

$$0.709 \dots < r < 1,$$

and establish the exact value of the constant  $a_s$ .

**Theorem.** *Every arc of the level line  $L(f, r)$ ,  $f \in S$ ,  $r_s < r < 1$ , lying in the annulus*

$$\lambda \underline{R} \bar{R} \leq |w| \leq \bar{R},$$

where

$$\lambda = \frac{5\sqrt{5}}{4} e^{-4 \operatorname{arcctg} 2},$$

is starlike.

In the case where  $r_0 < r < 1$ , where  $r_0$  is the unique root of the equation

$$\ln \frac{\sqrt{5}y}{2a(1-y^2)^2} + 2 \ln (\sqrt{1-a^2y^2} + \sqrt{y^2-a^2}) + \operatorname{arc} \sin \frac{1+y^2}{2\sqrt{2}y} - 3 \operatorname{arc} \operatorname{tg} 3 = 0, \quad a = \sqrt{2}-1, \quad (1)$$

there exists a function  $f(z) \in S$  whose level line  $L(f, r)$ , at some point

$$w_1 = f(z_1), \quad |z_1| = r,$$

of the wider annulus

$$(\lambda - \varepsilon) \underline{R} \bar{R} \leq |w| \leq \bar{R}, \quad \varepsilon > 0,$$

is not starlike.

**Proof.** 1°. Denote

$$zf'(z)/f(z) = F(z)$$

and establish an upper estimate for the real functional prescribed on the class  $S$

$$J_1(f) = \ln |f(z_0)| - 2 \operatorname{arg} F(z_0) \quad (2)$$

\* In all cases, when speaking of the starlikeness of an analytic curve, we mean its starlikeness with respect to the origin.

for fixed  $z_0$  on the circle  $|z| = r$ ,  $a < r < 1$ . To this end, let us consider the functional (2) on the subclass  $S(k)$ , dense in  $S$ , of functions determined by integrals of the Löwner equation

$$\frac{\partial f(z, t)}{\partial t} = -f(z, t) \frac{1 + k(t)f(z, t)}{1 - k(t)f(z, t)}, \quad 0 \leq t \leq \infty,$$

with a piecewise-continuous characteristic function  $k(t)$ ,  $|k(t)| = 1$ . Since on  $S(k)$  \*

$$\ln |f(z_0)| = \ln \frac{r}{1-r^2} - 2 \int_0^r \cos \psi \frac{ds}{1-s^2}, \quad \arg \frac{z_0 f'(z_0)}{f(z_0)} = - \int_0^r \Phi(\psi, s) \frac{ds}{1-s^2},$$

where

$$\Phi(\psi, s) = \frac{2(1-s^2) \sin \psi}{1-2s \cos \psi + s^2}$$

and  $\psi = \psi(s)$  is some real piecewise-continuous function on the segment  $0 \leq s \leq r$ , it follows that on the subclass  $S(k)$

$$J_1(f) = \ln \frac{r}{1-r^2} + 2 \int_0^r G(\psi, s) \frac{ds}{1-s^2}, \quad (3)$$

where here  $G(\psi, s) = \Phi(\psi, s) - \cos \psi$ . The greatest value of  $G(\psi, s)$ , if the second argument is regarded as fixed, is attained at the point  $\psi = \varphi$ ,  $0 \leq \varphi \leq \pi$ , which is among the solutions of the equation

$$\frac{2(1-s^4) \cos \varphi + 4s^3 - 4s}{(1-2s \cos \varphi + s^2)^2} + \sin \varphi = 0.$$

It is easy to verify that this equation decomposes into the following two equations:

$$s^2 + 2s \sin \varphi - 1 = 0,$$

$$2s \cos^2 \varphi - 2(1+s^2) \cos \varphi - (1-s^2) \sin \varphi + 2s = 0. \quad (4)$$

The solutions of the first of them,

$$\varphi_1 = \arcsin \frac{1-s^2}{2s}, \quad \varphi_2 = \pi - \varphi_1,$$

which occur only for  $a \leq s \leq r$ , correspond to one and the same value of the function  $G(\psi, s)$ , equal to  $\frac{1+s^2}{2s}$ .

We shall seek the solution of the second equation in the form

$$\varphi_3 = 2 \operatorname{arctg} \left( x \frac{1-s}{1+s} \right), \quad 0 \leq s \leq r, \quad (5)$$

regarding here  $x$  as a new nonnegative parameter. As a result of substituting (5) into (4), we find the formula

$$\frac{1+s}{1-s} = \sqrt{\frac{x^4 - x^3}{x+1}}, \quad (6)$$

from which it follows that, for  $s > 0$ ,

$$x > \frac{1 + \sqrt{5}}{2}.$$

Since  $\frac{ds}{dx} > 0$ , the relation between the parameters  $x$  and  $s$  established by formula (6) is one-to-one, and therefore (5), with (6) taken into account, gives a solution of equation (4). To this solution there corresponds the value of  $G(\psi, s)$ , equal to

$$-\frac{x^2 - 6x - 1}{x^2 + 1}.$$

\* The derivation of the two formulas given below is contained in (2), p. 256;  $|f(z_0, t)| = s, \quad \arg[k(t)f(z_0, t)] =$

On the interval  $a < s \leq r$  the difference

$$G(\varphi_1, s) - G(\varphi_3, s) = \frac{2(x^2 - 2x - 1)^2}{(x^2 + 1)(x^2 - x - 1)}$$

is positive and, thus,

$$G(\varphi, s) = \begin{cases} G(\varphi_3, s), & \text{if } 0 \leq s \leq a, \\ G(\varphi_1, s) = G(\varphi_2, s), & \text{if } a < s \leq r. \end{cases}$$

After replacing  $G(\psi, s)$  in (3) by  $G(\varphi, s)$  and carrying out the integration, we obtain in  $S(k)$ , and hence also in  $S$ , the following estimate for the functional (2):

$$J_1(f) \leq \pi + \ln(\lambda R \bar{R}).$$

2°. By means of inessential changes in the preceding arguments one can establish that the functional

$$J_2(f) = \ln |f(z_0)| + 2 \arg F(z_0), \quad f \in S,$$

is bounded above by the constant  $\pi + \ln(\lambda R\bar{R})$ . Thus, also for the functional

$$J(f) = \ln |f(z_0)| + 2 \arg F(z_0), \quad f \in S,$$

the estimate

$$I(f) \leq \pi + \ln(\lambda R\bar{R}), \quad (7)$$

holds, independent of  $\arg z_0$ . This estimate is sharp, since equality in it is attained for the function

$$f(z) = \lim_{t \rightarrow \infty} e^t f(z, t),$$

where  $f(z, t)$  is the solution of the Löwner equation with characteristic function  $k(t)$ , constructed with account of the initial condition  $f(z_0, 0) = z_0$ , from the function

$$\psi(s, y) = \begin{cases} \varphi_3(s), & \text{if } 0 \leq s \leq a, \\ \varphi_1(s), & \text{if } a < s \leq y \quad (a \leq y \leq r), \\ \varphi_2(s), & \text{if } y < s \leq r. \end{cases}$$

3°. To finish the proof of the first part of the theorem, let us note that  $L(f, r)$  is star-shaped with respect to the point  $f(z_0)$  if and only if  $2|\arg F(z_0)| \leq \pi$ . Suppose now that  $\lambda R\bar{R} \leq |f(z_0)|$ . Hence, on the basis of inequality (7), we find that at the point  $f(z_0)$ ,  $2|\arg F(z_0)| \leq \pi$ .

4°. We proceed to the proof of the second part of the theorem. For the family of extremal functions giving equality in estimate (7), by direct computation we establish that

$$\arg F(z_0) = - \int_0^r \Phi(\psi(s, y), s) \frac{ds}{1-s^2} = -\frac{1}{2} Q(r, y),$$

where

$$\begin{aligned} Q(r, y) &= \ln \frac{\sqrt{10} r}{1-r^2} - h(a) - h(r) + 2h(y) + 3 \operatorname{arc} \operatorname{ctg} 3, \\ h(s) &= 2 \int \cos \varphi_1 \frac{ds}{1-s^2} = \\ &= \operatorname{arcsin} \frac{1+s^2}{2\sqrt{2}s} + 2 \ln(\sqrt{1-a^2s^2} + \sqrt{s^2-a^2}) - \ln(1-s^2) + \operatorname{const}. \end{aligned}$$

From the continuity and monotone increase of  $Q(y, y)$  on  $[a, 1]$  from  $Q(a, a) < \pi$  to  $Q(1, 1) = +\infty$ , there follows the existence on  $(a, 1)$  of a unique solution  $r_0$  of the equation  $Q(y, y) = \pi$ , i.e., of equation (1). Fix in the interval  $(r_0, 1)$  some point  $\rho$ . The function  $Q(\rho, y)$  is continuous and increases monotonically for  $a \leq y \leq \rho$ , from  $Q(\rho, a) < Q(1, a) < \pi$  to  $Q(\rho, \rho) > \pi$ . Therefore, for any number  $\delta > 0$ ,  $\delta \leq Q(\rho, \rho) - \pi$ , there will be found such a value

value  $y_0 = y_0(\delta)$  such that  $Q(\rho, y_0) = \pi + \delta$ , i.e.  $-2 \arg F(z_0) = \pi + \delta$ . Taking this equality into account, from (7) we obtain that

$$|f(z_0)| = \lambda e^{-\delta} R \bar{R}.$$

Thus, in the class  $S$  there exists a function  $f(z)$ , depending on the choice of  $z_0$  and  $\delta$ , with a level line  $L(f, r)$  having a non-star-shaped arc outside the annulus

$$\lambda R \bar{R} \leq |w| \leq R.$$

The proof of the theorem is complete.

**Remark.** Computations show that  $\lambda = 0.437 \dots < 7/16$  ( $\lambda - 7/16 = O(10^{-5})$ ), and  $r_0 = 0.709 \dots$

**Corollary 1.** If  $r \in (r_s, r_0)$ , then

$$a_s(r) \leq \lambda \frac{r}{(1+r)^2};$$

if  $r \in (r_0, 1)$ , then

$$a_s(r) = \lambda \frac{r}{(1+r)^2}.$$

**Corollary 2.**  $a_s = a_s(1) = \lambda/4 = 0.109 \dots$

**Corollary 3.** If  $f(z) \in S$ , then an arc of the level line  $L(f, r)$ ,  $r_s < r < 1$ , lying in the annulus

$$\frac{7r^2}{16(1-r^2)^2} \leq |w| \leq \frac{r}{(1-r)^2},$$

is star-shaped with respect to the origin.

**Corollary 4.** For the subclass  $S_p \subset S$  ( $p = 1, 2, \dots$ ,  $S_1 \equiv S$ ) of  $p$ -symmetric functions  $f(z)$ , the arcs of the level line  $L(f, r)$ ,  $r_{ps} = \sqrt[p]{r_s} < r < 1$ , are star-shaped in the annulus

$$\frac{\sqrt[p]{\lambda} r^2}{(1-r^{2p})^{2/p}} \leq |w| \leq \frac{r}{(1-r^p)^{2/p}}. \quad (8)$$

**Corollary 5.** For the class  $\Sigma_{0p}$  of functions

$$F(z) = \frac{1}{f(1/z)}, \quad f(z) \in S_p,$$

the arcs of the level line  $L(F, r)$ ,  $1 < r < R_{ps} = r_{ps}^{-1}$ , are star-shaped in the annulus

$$\frac{(r^p - 1)^{2/p}}{r} \leq |w| \leq \frac{(r^{2p} - 1)^{2/p}}{\sqrt[p]{\lambda} r^2}. \quad (9)$$

**Remark.** In Corollaries 4 and 5, for  $\sqrt[p]{r_0} < r < 1$  (respectively  $1 < r < \sqrt[p]{r_0^{-1}}$ ) one cannot replace  $\lambda$  by a smaller number without violating the indicated properties of the annuli (8) and (9).

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## REFERENCES

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*Note: Figure translations are in progress. See original paper for figures.*

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