MATHEMATICAL PHYSICS

M. V. MASLENNIKOV, Yu. S. SIGOV

1964

SovietRxiv

Abstract

Full Text

MATHEMATICAL PHYSICS

M. V. MASLENNIKOV, Yu. S. SIGOV

A DISCRETE MODEL OF MATTER IN THE PROBLEM OF FLOW AROUND BODIES BY A RAREFIED PLASMA

(Presented by Academician M. V. Keldysh, June 10, 1964)

In many applied problems it is essential to know the detailed structure of the disturbance region around a body moving in a rarefied plasma. Since the equations describing the collective interaction of charged particles in this region are essentially nonlinear, satisfactory results can be obtained analytically only in a few special cases. In this connection, the development of effective numerical methods for solving such problems is of interest—methods not associated with rigid restrictions on the geometry and physical formulation. Apparently, discrete modeling of a rarefied plasma is especially promising. In the present work this method is used to solve the following problem.

1°. An axisymmetric body A is placed in a stationary stream of plasma, neutral at infinity, moving parallel to the axis A with macroscopic velocity $\mathbf{v}$. It is required to find the self-consistent electrostatic potential $\varphi(\mathbf{r})$ established in the neighborhood of A. The function $\varphi(\mathbf{r})$ depends on the characteristics of the incident plasma stream, on the dimensions and shape of A, and on the potential $\varphi_\Sigma(\mathbf{r})$ prescribed on its surface Σ. For definiteness we shall assume that an ion striking Σ is absorbed and that in the neighborhood of A there are no ions moving along finite trajectories. The spatial charge density is the sum of the electron and ion densities:
$\rho(\mathbf{r}) = \rho^E(\mathbf{r}) + \rho^I(\mathbf{r})$. We shall assume that $\rho^E(\mathbf{r})$ is uniquely determined by the specification of the self-consistent potential φ and of ρ^I at the same point $\mathbf{r}$: $\rho^E(\mathbf{r}) = F(\varphi(\mathbf{r}), \rho^I(\mathbf{r}))$. Such an assumption is admissible if the speed $v = |\mathbf{v}|$ is small in comparison with the mean thermal speed of the electrons. For example, in the simplest case one may assume that the electrons are in thermal equilibrium with $\varphi(\mathbf{r})$:

$$\rho^E(\mathbf{r}) = \rho_0 \exp\left(\frac{\varepsilon\varphi(\mathbf{r})}{kT_e}\right) \tag{1}$$

($\rho_0 = \text{const}$, k is Boltzmann' s constant, T_e is the effective temperature of the electron gas, $-\varepsilon$ is the electron charge).

Assuming that the characteristic dimensions of the neighborhood Ω of the body A that interests us are small in comparison with the mean free path of the ions, we shall describe the motion of the ion component by means of the collisionless kinetic Vlasov equation. The characteristics of this equation coincide with the trajectories of individual ions moving in the field of the potential $\varphi(\mathbf{r})$. To construct these characteristics we introduce a discrete model of the plasma, assigning to groups of ions located in small nonintersecting volumes of phase space material points carrying the total masses and total charges of the corresponding groups. The motion of these points will be considered by the methods of classical mechanics. This calculation gives us the desired characteristics of the Vlasov equation and makes it possible to compute $\rho^I(\mathbf{r})$.

2°. The problem is considered in cylindrical coordinates (r, z). The axis Oz coincides with the axis A and is oriented along the velocity $\mathbf{v}$. The region Ω, in which all calculations are carried out, surrounds A and is an elongated-

a cylinder extended along Oz, with circular faces $T^{\mp}$, corresponding to the coordinates $z = z^-$ and $z = z^+ > z^-$. The disk T^- is divided into narrow concentric rings with mean radii b_k, $b_0 < b_1 < \cdots < b_N$, and areas s_k, $k = 1, 2, \ldots, N$. From the points $\mathbf{r}_k^0 = (b_k, z^-)$ of the face T^- the characteristics $\mathbf{r} = \mathbf{r}_k(t)$ of the Vlasov equation are launched. The equations of these characteristics are Newton' s equations:

$$\ddot{\mathbf{r}}_k = -\frac{Ze}{M}\nabla\varphi(\mathbf{r}_k) \tag{2}$$

with initial conditions:

$$\mathbf{r}_k(0) = \mathbf{r}_k^0, \qquad \dot{\mathbf{r}}_k(0) = \mathbf{v}, \qquad k = 1, 2, \ldots, N$$

(Ze is the ion charge, M its mass).

Introduce in the plane rOz a rectangular grid with cells Δ_{ij}, and let $t_{ij}^{(k)}$ be the measure of the set of those t for which $\mathbf{r}_k(t) \in \Delta_{ij}$. Then the value of ρ^I in the cell Δ_{ij} is

$$\rho_{ij}^I = n_0 Ze\, v\tau_{ij}^{-1} \sum_{k=1}^{N} s_k t_{ij}^{(k)} \tag{3}$$

$$\left(\tau_{ij} = 2\pi \int_{\Delta_{ij}} r\, dr\, dz, \quad n_0 \text{ is the spatial density of ions at infinity}\right).$$

Thus, (2) and (3) make it possible, for a given $\varphi(\mathbf{r})$, to find the corresponding $\rho^I(\mathbf{r})$: $\rho^I(\mathbf{r}) = \hat{A}\varphi(\mathbf{r})$, where $\hat{A}$ is a certain nonlinear operator.

If it is necessary to take into account the presence of different species of ions and their thermal motion, then from each point $\mathbf{r}_k^0$ a bundle of characteristics

$\mathbf{r} = \mathbf{r}_{kk'}(t)$, $k' = 1, 2, \dots, N'$, is launched. The characteristics of this bundle differ from one another only by the coefficients Ze/M of equation (2) (ions of different species) and by the initial condition $\dot{\mathbf{r}}_{kk'}(0) = \mathbf{v} + \mathbf{u}_{k'}$ ($\mathbf{u}_{k'}$ is the thermal velocity of the ion). In addition, the presence of azimuthal components in the thermal velocities of the ions leads to the appearance, on the right-hand side of (2), of a term describing the centrifugal forces acting on the ion. The form of this term is uniquely determined by the initial conditions of the corresponding characteristic.

The system of equations of the problem is closed by adjoining to (2) the quasi-linear Poisson equation

$$\Delta\varphi(\mathbf{r}) = -4\pi\left[\rho^I(\mathbf{r}) - F(\varphi(\mathbf{r}), \rho^I(\mathbf{r}))\right] \tag{4}$$

with boundary conditions: $\varphi(\mathbf{r}) = \varphi_\Sigma(\mathbf{r})$ for $\mathbf{r} \in \Sigma$; $\partial\varphi(\mathbf{r})/\partial r = 0$ for $r = 0$; and a suitable condition on the outer boundary Ω (for example, $\varphi = 0$). Equation (4) makes it possible, from a given distribution of the ion density $\rho^I(\mathbf{r})$, to determine the corresponding field $\varphi(\mathbf{r})$: $\varphi(\mathbf{r}) = \hat{B}\rho^I(\mathbf{r})$.

Together with the operators $\hat{A}$ and $\hat{B}$, introduce into the discussion one more "smoothing" operator $\hat{P}$, acting on the ion densities $\rho^I(\mathbf{r})$ according to the rule

$$\hat{P}\rho^I(\mathbf{r}) = \min\{q, \rho^I(\mathbf{r})\}, \qquad q = \text{const} > 0.$$

The role of this operator is as follows. Because of the weak stability of the solutions of equations (2), small changes in φ may correspond to sharp (in most cases unphysical) local perturbations of ρ^I. $\hat{P}$ makes it possible to carry out the simplest regularization of intermediate values of ρ^I.

3°. The problem (2), (4) is solved by the method of successive approximations (the so-called iterations over ion fluxes): choose some zeroth approximation $\varphi^{[0]}(\mathbf{r})$ to the self-consistent potential φ, and for $n \geqslant 0$ set

$$\varphi^{[n+1]} = \alpha\hat{B}\hat{P}\hat{A}\varphi^{[n]} + (1-\alpha)\varphi^{[n]} \tag{5}$$

(α is an interpolation parameter, $0 < \alpha \leqslant 1$). A variant of this method was apparently first used in (^{1}). The convergence of the successive

of approximations $\{\varphi^{[n]}\}$ means the existence and, in a certain sense, the stability of the stationary solution of the problem. A general investigation of this convergence is very difficult. However, in some simple cases it can be carried out and, if ρ^E is determined by formula (1), the following necessary condition for convergence can be obtained:

$$\frac{Mv^2}{2} > \vartheta k T_e Z, \tag{6}$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

where ϑ increases with increasing volume V_Ω of the region Ω, approaching (for $V_\Omega \gg D^3$, D the Debye radius) its asymptotic value $\vartheta_0 \sim 1$. If $\{\varphi^{[n]}\}$ converges, then its limit $\varphi^{[\infty]} = \varphi$ is accepted as the solution of the problem. In doing so, obviously, the condition $\hat{P}\hat{A}\varphi = \hat{A}\varphi$ must be satisfied. Otherwise the calculations must be repeated with a larger q.

Fig. 1. Equipotential curves ($\psi = e\varphi/kT_e = \text{const}$) according to the results of the first iteration by ion fluxes. The plasma moves to the right with velocity v

4°. The method described was programmed. Equation (4) was solved by the method of establishment: it was associated with the parabolic equation

$$\frac{\partial\varphi(x,\mathbf{r})}{\partial x} = \Delta\varphi(x,\mathbf{r}) + 4\pi\left[\rho^I(\mathbf{r}) - F\right] \tag{7}$$

and the limit of $\varphi(x,\mathbf{r})$ as $x \to \infty$ was taken as the solution of (4). To solve (7), the locally one-dimensional scheme of A. A. Samarskii (^{2}) was used, imposing minimal requirements on the shape of the region Ω. This made it possible to treat arbitrary axisymmetric bodies A in a uniform way.

Fig. 2. Equipotential curves of the self-consistent field for $M = 16$ a.m.u.

5°. As an example, the flow around a sphere of radius $R = 50$ cm with $\varphi_\Sigma = 0$ was considered. Plasma parameters: $T_e = 5000°\text{K}$, $T_i = 0$; Debye radius $D = R$; the ion mass was taken in turn to be 16 and 1 a.m.u.; flow velocity $v = 10^6$ cm/sec. The number of cells in the difference grid was ~ 900; in this case up to 3000 ion trajectories were "emitted" toward the body.

The pattern of sections of equipotential surfaces by the rOz plane gives an idea of the characteristic features of the disturbance zone (Figs. 2 and 3). Behind the body, for several Debye radii, there extends an electron

"wake" accompanied by a potential well. Its maximum depth is of the order of $0.5\,kT_e$. For comparison with the self-consistent solution, Fig. 1 gives equipotentials according to the results of the first iteration with respect to the ion fluxes, which corresponds to solution (4) with $\rho^I(\mathbf{r}) = \rho_0$ everywhere except the cylindrical region of the ion "shadow" behind the body, where $\rho^I(\mathbf{r}) = 0$.

Even the calculation of the first variants makes it possible to establish the presence of a peculiar effect— "stratification" of the plasma in the perturbation zone. In the region behind the body the plasma is polarized: positive and

Fig. 3. Equipotential curves of the self-consistent field for $M = 1$ a. e. m.

Figure 3: Fig. 3. Equipotential curves of the self-consistent field for $M = 1$ a. e. m.

negative charges alternate, decreasing in amplitude with distance from the body (Fig. 3).

Fig. 3. Equipotential curves of the self-consistent field for $M = 1$ a. e. m.

6°. The examples described above were calculated under the condition $\varphi_\Sigma =$ const. Let us note that, by varying φ_Σ and monitoring the total current to the body J_Σ, it is not difficult to satisfy another important condition: $J_\Sigma = 0$, and thus to determine the potential established on an isolated body in a plasma.

The method presented makes it possible to solve effectively problems of the theory of a moving probe and of the dynamic interaction of a plasma with a body moving in it.

The authors express their gratitude to T. A. Sushkevich for preparing the programs, and to K. I. Gringauz, M. L. Levin, and A. A. Samarskii for fruitful discussions.

Received
9 VI 1964

REFERENCES

1. A. N. Davis, I. Harris, *Proc. 2-nd Symp. on Rarefied Gas Dynamics*, 1960.

2. A. A. Samarskii, *Zhurn. vychislit. matem. i matem. fiz.*, **2**, No. 5, 787 (1962).

Note: Figure translations are in progress. See original paper for figures.