DESCRIPTION OF UNITARY REPRESENTATIONS OF THE LORENTZ GROUP IN A SPACE WITH INDEFINITE METRIC

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Abstract

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MATHEMATICS

R. S. ISMAGILOV

DESCRIPTION OF UNITARY REPRESENTATIONS OF THE LORENTZ GROUP IN A SPACE WITH INDEFINITE METRIC

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1. In the present note we give a description of an arbitrary representation of the group G of complex unimodular matrices of the second order in a space with indefinite metric (the theory of such spaces is set forth in ([1,2])). The problem of studying such representations was posed by M. A. Naimark; he also obtained a number of results on unitary representations in Π-spaces ([4,5]).

Irreducible representations of the group G that are unitary in a Π_n-space are described in ([3]). The study of an arbitrary unitary representation in a Π_n-space is complicated by the fact that for such representations there is no theorem on complete reducibility. We shall describe the "simplest"(after the irreducible ones) representations of the group G that are unitary in a Π_n-metric (representations of type T^k and S); any unitary representation turns out to be a direct orthogonal sum of a finite number of irreducible representations, a unitary (in the usual sense) representation, and a finite number of representations of type T^k and S' (some of these components may, of course, be absent).

We set forth the scheme for describing the representations. Let T_g be a representation of the group G that is unitary in the Π_n-space H. It turns out that it is sufficient to consider only those representations for which there exists in H an n-dimensional null (i.e. with identically zero scalar product) invariant subspace. More precisely, the following holds.

Theorem 1. Let D be the maximal null subspace in H invariant with respect to T_g (if there are no such subspaces, we put $D = 0$).

Then:

1) if $D = 0$, then there is a decomposition

$$T_g = T_g^{i_1} \oplus T_g^{i_2} \oplus \cdots \oplus T_g^{i_s} \oplus U_g, \qquad i_k > 0,$$

where $T_g^{i_k}$ is an irreducible representation unitary in the Π_{i_k}-metric, and U_g is a unitary representation (in the usual sense);

2) if $\dim D = k < n$, then

$$T_g = T_g^{j_1} \oplus \cdots \oplus T_g^{j_m} \oplus T_g^*,$$

where the representations $T_g^{j_k}$ are irreducible and unitary in the Π_{j_k}-metric ($j_k > 0$), while the representation T_g^* is unitary in the Π_k-metric; in addition, the subspace H^*, in which T_g^* acts, contains D, and the subspace D is a maximal null invariant subspace in H^*.

Thus, in what follows we may restrict ourselves to representations T_g satisfying the following condition.

A. In the Π_n-space H, where the representation T_g acts, there exists an n-dimensional null invariant subspace D.

Introduce two more subspaces in H: the subspace M of vectors $x \in H$ orthogonal to D, and a subspace $R \subset H$ such that $R \dotplus D = M$; it is obvious that the metric in R is positive.

Obviously, $T_g M = M$; denote by T_g^M the restriction of T_g to the subspace M. It turns out that the study of the representation T_g reduces to the study of the representation T_g^M; more precisely, the following Theorems 2 and 3 hold.

Theorem 2. Let T_g' and T_g'' be two representations satisfying condition A, and let D be the maximal n-dimensional null invariant subspace for T_g' and T_g''. Suppose, furthermore, that $T_g'^M = T_g''^M$ (i.e. $T_g' x = T_g'' x$ for all $x \in M$). Then the representations T_g' and T_g'' are unitarily equivalent.

Theorem 3. Let T_g satisfy condition A. Let

$$M = M_1 \dotplus M_2,$$

where M_1 and M_2 are orthogonal in the sense of the metric (x, y) and invariant with respect to T_g. Then there exists a decomposition

$$H = H_1 \oplus H_2,$$

such that:

$$1) \quad T_g H_i = H_i \quad (i = 1, 2),$$
$$2) \quad M_i \subset H_i \quad (i = 1, 2).$$

The representation T_g^M is a coupling of an n-dimensional representation with a unitary one. Let us study such representations in greater detail.

2. Let M be a Hilbert space with scalar product $\{x, y\}$, $M = D \oplus R$, $\dim D = n < \infty$. Let (x, y) denote the degenerate scalar product in M defined as follows: $(x, y) = \{x, y\}$ if $x, y \in R$; $(x, y) = 0$ if $x \in D$ or $y \in D$. Consider a representation T_g of the group G in M, preserving the scalar product (x, y), (i.e. $(T_g x, T_g y) = (x, y)$, $x, y \in M$); obviously, $T_g D = D$, and for $x \in R$

$$T_g x = V_g x + A_g x, \tag{1}$$

where V_g is a unitary representation of G in R, while A_g maps R into D. Let us consider one special class of such representations.

Definition 1. We shall say that a representation T_g, acting in M and possessing the properties described above, is a **representation of type** T_k, if: 1) the restriction S_g of the representation T_g to D is a multiple of the irreducible (finite-dimensional) representation $S_{n,n}$, while V_g is a multiple* of the unitary irreducible representation $T_{n,-n}$, which is the nearest relative of the representation $S_{n,n}$ ([3], p. 212); 2) the space M cannot be decomposed into a direct sum of two subspaces invariant with respect to T_g and orthogonal in the sense of the bilinear form (x, y).

From condition 2) it follows, in particular, that T_g contains no unitary parts, i.e. in M there do not exist invariant subspaces on which the form (x, y) is nondegenerate (and hence positive definite).

Definition 1′. A representation T_g, acting in M, will be called a **representation of type** S, if: 1) $S_g = E$ (i.e. $T_g x = x$, $x \in D$); 2) $V_g = V'_g \oplus V''_g$, where V'_g is a multiple of the representation $T_{1,-1}$ (the nearest relative of the identity representation), and the representation V''_g decomposes only into representations of the supplementary series; 3) the restriction of the representation T_g to $H(V'_g) \oplus D$ (here $H(V'_g)$ denotes that subspace of the space R on which V'_g acts) is a representation of type T_1.

It turns out that from unitary (in the usual sense) representations, finite-dimensional representations, and representations of types T_k and S (see the definition), one can compose any representation possessing the properties listed at the beginning of this section. More precisely, the decomposition

$$T_g = U_g \dotplus L_g \dotplus \sum_1^s T_g^k \dotplus T'_g,$$

holds

* The multiplicity may be infinite.

where U_g is a unitary representation, L_g is a finite-dimensional representation; the representation T_g^k belongs to type T_k, and the representation T'_g to type S_1; moreover, the subspaces on which these components of the representation T_g act are mutually orthogonal in the sense of the form (x, y). We note that L_g includes, in particular, all those finite-dimensional subrepresentations that are not relatives of any unitary representations.

3. It is now easy to describe the representations that are unitary in the Π_n-metric.

Definition 2. Let T_g be a representation of a group G that is unitary in the Π_n-metric. We shall call T_g a representation of type T^k if in the representation space there exists an n-dimensional null invariant subspace D, and the restriction T_g^M of the representation T_g to the subspace $M = \{x : x \perp D\}$ is a representation of type T_k. A representation of type S' is defined analogously. We now formulate the main theorem.

Theorem 4. *A representation T_g of a group G that is unitary in the Π_n-metric admits a decomposition into a direct orthogonal sum*

$$T_g = L_g \oplus U_g \oplus \sum_{i=1}^{s} \oplus T_g^{k_i} \oplus \sum_{j=1}^{k} \oplus \widetilde{T}_g^{p_j} \oplus \sum_{s=1}^{l} \oplus T_g^{q_s},$$

where L_g is finite-dimensional, U_g is a unitary (in the ordinary sense) representation; $T_g^{k_i}$ belongs to type T^{k_i}, $\widetilde{T}_g^{p_j}$ to type S'; $T_g^{q_s}$ is irreducible and unitary in the Π_{q_s}-metric ($q_s > 0$).

4. Theorem 2 asserts that a representation T_g that is unitary in the Π_n-metric is determined, up to unitary equivalence, by its part T_g^M; the following question arises: let D be an n-dimensional null subspace of a Π_n-space H, $M = \{x : x \perp D\}$, and let T_g^M be a representation of the group G in M preserving the bilinear form (x, y) (i.e. $(T_{gx}, T_{gy}) = (x, y)$, $x, y \in M$). Can the representation T_g^M be extended to a unitary representation T_g in the whole space H?

The answer to this question is positive if the representation V_g that was constructed by formula (1) decomposes into a finite number of irreducible representations. The general case remains unclear.

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Voronezh State University

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