THE EXACT VALUE OF THE PALEY-WIENER CONSTANT

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Abstract

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MATHEMATICS

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THE EXACT VALUE OF THE PALEY-WIENER CONSTANT

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A basis $\{e_k\}$ of a Hilbert space is called a **Riesz basis** if, for every element $x = \sum_{k=1}^{\infty} a_k e_k$, the inequality

$$A\left(\sum |a_k|^2\right)^{1/2} \leqslant \|x\| \leqslant B\left(\sum |a_k|^2\right)^{1/2} \qquad (0 < A \leqslant B < \infty)$$

holds. Let $\lambda_k = k + \delta_k$ $(k = 0, \pm 1, \pm 2, \ldots)$ be real numbers satisfying the condition

$$\sup_k |\delta_k| = d < D. \tag{1}$$

Paley and Wiener ([1]) showed that if $D = 1/\pi^2$, then the sequence $\{e^{i\lambda_k t}\}_{-\infty}^{\infty}$, where the λ_k are subject to condition (1), is a Riesz basis in the space $L_2(-\pi, \pi)$. Duffin and Eachus ([2]) (see also ([3])) established that this result is valid for $D = \ln 2/\pi \approx 0.22$, and V. D. Golovin ([4]) raised the value of D to 0.24. According to a theorem of Levinson ([5]), for $D > 0.25$ the assertion ceases to be true. In the present note we shall show that the exact boundary of admissible D' s is $D = 0.25$. As in the papers ([1–4]), the starting point of our considerations will be the

Paley-Wiener Lemma. *If the system $\{e^{i\lambda_k t}\}$ is close to the system $\{e^{ikt}\}$ in the sense that*

$$\left\|\sum_k a_k e^{ikt} - \sum_k a_k e^{i\lambda_k t}\right\| \leqslant \theta \left\|\sum_k a_k e^{ikt}\right\| = \theta\left(\sum_k |a_k|^2\right)^{1/2}$$

for some $\theta < 1$ and all finite sets of numbers a_k, then the system $\{e^{i\lambda_k t}\}$ is a Riesz basis in $L_2(-\pi, \pi)$.

Theorem 1. *If the sequence $\lambda_k = k + \delta_k$ is subject to the condition*

$$\sup_k |\delta_k| = d < 0.25 \quad (k = 0, \pm 1, \pm 2, \ldots),$$

then the system $\{e^{i\lambda_k t}\}$ *is a Riesz basis in* $L_2(-\pi,\pi)$.

Proof. According to the Paley-Wiener lemma, the question reduces to the investigation of the upper bound of the expression

$$U=\left\|\sum_k a_k(1-e^{i\delta_k t})e^{ikt}\right\|=\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_k a_k(1-e^{i\delta_k t})e^{ikt}\right|^2 dt\right\}^{1/2}, \qquad (2)$$

taken over all finite sets of numbers a_k such that $\sum_k |a_k|^2 \leqslant 1$. If this upper bound is less than one, then the system $\{e^{i\lambda_k t}\}$ is a Riesz basis.

Expand the function $\psi(t) = 1 - e^{i\delta t}$ $(-\pi \leqslant t \leqslant \pi)$ in a Fourier series with respect to the orthogonal system $\{1; \cos\nu t; \sin(\nu-\frac{1}{2})t\}$ $(\nu = 1, 2, ...)$:

$$1-e^{i\delta t}=\left(1-\frac{\sin\pi\delta}{\pi\delta}\right)+\sum_{\nu=1}^{\infty}\frac{(-1)^\nu 2\delta\sin\pi\delta}{\pi(\nu^2-\delta^2)}\cos\nu t+$$
$$+\,i\sum_{\nu=1}^{\infty}\frac{(-1)^\nu 2\delta\cos\pi\delta}{\pi[(\nu-\frac{1}{2})^2-\delta^2]}\sin(\nu-\tfrac{1}{2})t. \qquad (3)$$

Substitute (3) into (2) and change the order of summation:

$$U=\Bigg\|\sum_k\left(1-\frac{\sin\pi\delta_k}{\pi\delta_k}\right)a_k e^{ikt}+\sum_{\nu=1}^{\infty}\cos\nu t\sum_k\frac{(-1)^\nu 2\delta_k\sin\pi\delta_k}{\pi(\nu^2-\delta_k^2)}a_k e^{ikt}$$
$$+\,i\sum_{\nu=1}^{\infty}\sin(\nu-\tfrac{1}{2})t\sum_k\frac{(-1)^\nu 2\delta_k\cos\pi\delta_k}{\pi[(\nu-\frac{1}{2})^2-\delta_k^2]}a_k e^{ikt}\Bigg\|.$$

Apply the triangle inequality:

$$U\leqslant\left\|\sum_k\left(1-\frac{\sin\pi\delta_k}{\pi\delta_k}\right)a_k e^{ikt}\right\|+\sum_{\nu=1}^{\infty}\left\|\cos\nu t\sum_k\frac{(-1)^\nu 2\delta_k\sin\pi\delta_k}{\pi(\nu^2-\delta_k^2)}a_k e^{ikt}\right\|$$
$$+\sum_{\nu=1}^{\infty}\left\|\sin(\nu-\tfrac{1}{2})t\sum_k\frac{(-1)^\nu 2\delta_k\cos\pi\delta_k}{\pi[(\nu-\frac{1}{2})^2-\delta_k^2]}a_k e^{ikt}\right\|.$$

Estimate each term:

$$\left\|\sum_k\left(1-\frac{\sin\pi\delta_k}{\pi\delta_k}\right)a_k e^{ikt}\right\|\leqslant\left(1-\frac{\sin\pi d}{\pi d}\right)\left\|\sum_k a_k e^{ikt}\right\|,$$

$$\left\|\cos \nu t \sum_k \frac{(-1)^\nu 2\delta_k \sin \pi\delta_k}{\pi(\nu^2-\delta_k^2)} a_k e^{ikt}\right\| \leqslant \frac{2d \sin \pi d}{\pi(\nu^2-d^2)} \left\|\sum_k a_k e^{ikt}\right\|,$$

$$\left\|\sin(\nu-\tfrac{1}{2})t \sum_k \frac{(-1)^\nu 2\delta_k \cos \pi\delta_k}{\pi[(\nu-\frac{1}{2})^2-\delta_k^2]} a_k e^{ikt}\right\| \leqslant \frac{2d \cos \pi d}{\pi[(\nu-\frac{1}{2})^2-d^2]} \left\|\sum_k a_k e^{ikt}\right\|.$$

Thus:

$$U \leqslant \left\{1-\frac{\sin \pi d}{\pi d} + \sin \pi d \sum_{\nu=1}^{\infty} \frac{2d}{\pi(\nu^2-d^2)} + \cos \pi d \sum_{\nu=1}^{\infty} \frac{2d}{\pi[(\nu-\frac{1}{2})^2-d^2]}\right\} \left\|\sum_k a_k e^{ikt}\right\|$$
$$=\left\{1-\frac{\sin \pi d}{\pi d} + \sin \pi d \left(\frac{1}{\pi d} - \operatorname{ctg} \pi d\right) + \cos \pi d \cdot \operatorname{tg} \pi d\right\} \left(\sum_k |a_k|^2\right)^{1/2}$$
$$=(1-\cos \pi d + \sin \pi d) \left(\sum_k |a_k|^2\right)^{1/2}.$$

Thus, the required upper bound of expression (2), for any $d < 0.25$, is strictly less than unity, which proves the theorem.

Duffin and Schaeffer [6] proved the following proposition:

Duffin-Schaeffer Theorem. If the system $\{e^{i\lambda_k t}\}_{-\infty}^{\infty}$ is a Riesz basis in $L_2(-\pi, \pi)$, and the real numbers μ_k satisfy the condition $\sup_k |\mu_k| < \infty$, then the system $\{e^{(\mu_k + i\lambda_k)t}\}$ is also a Riesz basis.

From this proposition and Theorem 1 there follows directly

Theorem 2. If the numbers z_k are such that

$$\sup_k |\operatorname{Im}(z_k - ik)| < 0.25; \qquad \sup_k |\operatorname{Re} z_k| < \infty,$$

then the system $\{e^{z_k t}\}_{-\infty}^{\infty}$ is a Riesz basis in $L_2(-\pi, \pi)$.

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Note: Figure translations are in progress. See original paper for figures.