MATHEMATICS

1964

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Abstract

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MATHEMATICS

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ON THE POTENTIAL OF A DOUBLE LAYER IN MULTIDIMENSIONAL SPACE

(Presented by Academician V. I. Smirnov on 21 IX 1964)

In the present note we consider potentials of a double layer situated on a general hypersurface of a multidimensional Euclidean space. In particular, necessary and sufficient conditions for the continuous extendability of such potentials are indicated here.

Let E_{n+1} be an oriented $(n+1)$-dimensional Euclidean space,

$$E = \{x;\ x \in E_{n+1},\ |x| = 1\}.$$

We regard the hypersphere E as positively oriented. By the symbol $H_p M$ we shall denote the p-dimensional Hausdorff measure of a set $M \subset E_{n+1}$. Throughout the article S will denote a closed hypersurface in E_{n+1}, on which we choose a positive orientation. (Thus, the order of points interior with respect to S is equal to one.) For simplicity we shall assume that $H_{n+1} S = 0$. If $x \in E_{n+1}$, then on $S \setminus \{x\} = S(x)$ we define the mapping P_x by setting

$$P_x(y) = \frac{y - x}{|y - x|}, \qquad y \in S(x).$$

For $U \subset S$ denote by bU the boundary of the set U relative to S. Let $B(x)$ be the system of all $U \subset S$ for which $\overline{U} \subset S(x)$ and $P_x(bU)$ is contained in a union of a finite number of hyperplanes passing through the center of the hypersphere E. If $U \in B(x)$, then for each point $z \in E \setminus P_x(bU)$ one can define the order $d(z, P_x, U)$ of the point z with respect to the mapping P_x, considered only on U (cf. ([5])); the function $d(z, P_x, U)$ of the variable z is defined almost everywhere (H_n) on E and is integrable (H_n). Put

$$m_x(U) = \int_E d(z, P_x, U)\, dH_n(z), \qquad U \in B(x).$$

The quantity $m_x(U)$ may be regarded as a measure of the solid angle under which U is seen from the point x. It is an additive set function of $U \in B(x)$. In order that it be possible to extend it to a completely finite generalized Borel measure on $S(x)$ (which we shall again denote by m_x), it is necessary and sufficient that

$$v(x) = \sup \sum_{j=1}^{q} |m_x(U_j)| < \infty,$$

where the supremum is taken over all finite systems of nonintersecting sets $U_1, \dots, U_q \in B(x)$. From the theory of continuous mappings (see ([7])) one can obtain an explicit representation for $v(x)$. The point of intersection of the hypersurface S with the half-line

$$L_x(z) = \{x + zr;\ r < 0\} \quad (z \in E)$$

will be called essential if it possesses arbitrarily small neighborhoods U on S satisfying the conditions

$$z \notin P_x(bU), \qquad d(z, P_x, U) \neq 0.$$

The number of all essential points of intersection of S with $L_x(z)$, whose distance from x is less than $R > 0$, will be denoted by $v_R(z, x)$ $(0 \le v_R(z, x) \le \infty)$.

Then $v_R(z, x)$ is a measurable (H_n) function of the variable z on E, whence one may set

$$v_R(x) = \int_E v_R(z, x)\, dH_n(z).$$

If R is so large that S is contained inside the sphere of radius R with center x, then $v_R(x) = v(x)$.

If $v(x) < \infty$, then for every bounded Baire function f on S there exists the (finite) integral

$$W(x, f) = \int_{S(x)} f\, dm_x,$$

which it is natural to call the **value at the point** x of the double-layer potential with density f.

Let us first clarify the relation between $v(x)$ and the area of the hypersurface S. By the area $p(S)$ of the hypersurface S we here mean its integral-geometric area in the sense of the definition proposed in ([2]) (cf. ([3]), p. 472).

Theorem 1. *Suppose that the points $x_1, \dots, x_{n+2}$ of the space E_{n+1} do not lie on one hyperplane. If $v(x_1) + \dots + v(x_{n+2}) < \infty$, then $p(S) < \infty$. Conversely, if $p(S) < \infty$, then for every point x at distance $\rho(x) = \rho > 0$ from the hypersurface S the estimate*

$$v(x) \leqslant \rho^{-n} p(S).$$

holds.

It is seen from this that, when considering the potentials $W(x, f)$ for $x \in E_{n+1} \setminus S$, it is natural to require that $p(S) < \infty$. In what follows we always assume that this condition is satisfied. For every bounded Baire function f, $W(x,f)$ is a harmonic function of the variable x on $E_{n+1}\setminus S$. Let us proceed to consider the behavior of the potential $W(x,f)$ near S. Denote by D_1 (respectively D_2) the domain interior (respectively exterior) with respect to S. If $y \in S$ and $v(y) < \infty$, then the set D_k has a definite density $\delta_k(y)$ at the point y ($k = 1, 2$), and we put

$$\varepsilon_1(y) = \delta_2(y)H_{nE}, \qquad \varepsilon_2(y) = -\delta_1(y)H_{nE}.$$

Theorem 2. *$y \in S$. If for every continuous function f on S the relation*

$$\infty > \limsup_{x\to y} |W(x,f)|, \quad x \in D_k,$$

holds, then there exists in S a neighborhood U of the point y such that

$$\sup_{x\in U} v(x) < \infty. \tag{1}$$

Conversely, if for some neighborhood U of the point y in S (1) holds, then $v(x)$ is bounded in some spatial neighborhood of the point y, and for every continuous function f on S there exist the limits

$$W_k(y,f) = \lim_{x\to y} W(x,f), \qquad x \in D_k \quad (k = 1, 2)$$

and the equalities

$$W_k(y,f) = W(y,f) + \varepsilon_k(y)f(y), \qquad k = 1, 2. \tag{2}$$

hold.

It follows from this that the condition

$$\sup_{y\in S} v(y) < \infty \tag{3}$$

is necessary and sufficient in order that, for every continuous function f on S, the potential $W(x,f)$ admit a continuous extension from D_k to $\overline{D}_k$.

It is now possible to apply the well-known Fredholm method for solving the first boundary-value problem of potential theory. In this connection it is useful to find an expression for the Fredholm radius of the operator

$$T_k f(y) = W_k(y,f) + \frac{1}{2}(-1)^k f(y)\, H_{nE},$$

acting on the space $C(S)$ of all continuous functions f on S with norm $\|f\| = \max_{y\in S} |f(y)|$. The reciprocal ωT_k of this radius is defined by the equality

$$\omega T_k = \inf_V \|T_k - V\|,$$

where the lower bound is taken over all completely continuous operators V acting on $C(S)$.

Theorem 3. The following equality holds:

$$\omega T_k = \lim_{R\to 0+} \sup_{y\in S} \left(v_R(y) + \left| \frac{1}{2} - \delta_k(y) \right| H_{nE} \right). \tag{4}$$

Let us note that

$$\left| \frac{1}{2} - \delta_1(y) \right| = \left| \frac{1}{2} - \delta_2(y) \right|.$$

For estimating $v_R(y)$ one may use the somewhat simpler function

$$\overline{v}_R(y) = \int_E \overline{v}_R(z, y)\, dH_n(z),$$

where $\overline{v}_R(z, y)$ is the number of all points of intersection of S with the segment $\{y + rz,\ 0 < r < R\}$ $(z \in E)$.

Obviously, $v_R(y) \le \overline{v}_R(y)$. In the plane case $(n = 1)$ equality holds here. In the general case $n > 1$ it may happen that $v_R(y) < \overline{v}_R(y)$. If S is a curve of bounded rotation in the plane $(n = 1)$, then, by Radon' s theorem, in the right-hand side of equality (4) one may omit the term $v_R(y)$ and the sign of passage to the limit (cf. ([6])). It is interesting to note that, generally speaking, this can no longer be done if S satisfies only the weaker assumption (3).

The theorems indicated are a generalization of the results of the note ([4]), which concerns the logarithmic potential. A communication ([1]) is devoted to the consideration of double-layer potentials in three-dimensional space.

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Received
12 IX 1964

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.