Mathematics

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Abstract

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INFINITESIMAL BENDINGS OF ONE CLASS OF RIBBED CYLINDROIDS

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1. We shall agree to call a surface C a **ribbed cylindroid** if it consists of n regular parts of zero Gaussian curvature and if: 1) C is homeomorphic to a cylindrical belt; 2) the boundary of C consists of two closed piecewise-smooth curves L_1 and L_2, lying in parallel planes and containing no rectilinear segments; 3) the lines of gluing σ_i $(i = 1, 2, \dots, n)$ of the regular parts, the ribs of the surface C, are at the same time generators of this surface.

We shall call a ribbed cylindroid C_A a **ribbed cylindroid of type** A if its tangent planes form, along the ribs σ_i, angles $0 < \theta_i < \pi$. Thus the surface C_A is nonregular, developable in the sense of Lebesgue and, on the whole, nonconvex.

We shall consider infinitesimal bendings of the surface C_A under which its ribs move as a rigid body; such infinitesimal bendings of the surface C_A will be called **admissible**.

For the surface C_A the following holds.

Theorem. *Under an admissible infinitesimal bending of the ribbed cylindroid C_A, on its boundaries L_1 and L_2 there are pairs of points, arbitrarily close to one another, the distances between which increase.*

In the proof of this theorem we shall use the lemma (see below) concerning infinitesimal bendings of plane curves.

2. Let a plane developable curve L_A, homeomorphic to a circle, be composed of a finite number of smooth convex arcs $l_1, l_2, \dots, l_n$ (the curve L as a whole being nonconvex), containing no rectilinear segments and having distinct tangents at their common points. For an infinitesimal bending of any arc l_i $(i = 1, 2, \dots, n)$ of such a curve L_A, a theorem of N. V. Efimov holds ([1]), and for the curves L_A defined above the following holds.

Lemma. *If, under an infinitesimal bending of the curve L_A, the variation of any of its smooth arcs is equal to zero, and the curvature does not decrease on any of the arcs l_i, then the velocity field of this bending is composed of a trivial summand and a field orthogonal to the plane of the curve L_A.*

Proof. Without loss of generality of the conclusions, we shall consider the curve L_A composed of two arcs l_1 and l_2. The vector equation of the curve L_A is

$$\mathbf{x}(s) = \begin{cases} {}^1\mathbf{x}, & s_0 \leq s \leq s_1, \\ {}^2\mathbf{x}, & s_1 \leq s \leq s_2; \quad s_2 \equiv s_0 \end{cases} \tag{1}$$

(the initial point of the vector $\mathbf{x}(s)$ is in the plane of the curve L_A; s is the arc length of the curve L_A); in this case the end of the vector $\mathbf{x} = {}^i\mathbf{x}(s)$ describes the corresponding arc l_i, and the regular functions ${}^i\mathbf{x}(s)$ admit a regular continuation.

Obviously,

$${}^1\mathbf{x}(s_0) = {}^2\mathbf{x}(s_0), \qquad {}^1\mathbf{x}(s_1) = {}^2\mathbf{x}(s_1),$$

$$\left[{}^1\mathbf{x}'(s_0^+),\, {}^2\mathbf{x}'(s_2^-)\right] \neq 0, \qquad \left[{}^1\mathbf{x}'(s_1^-),\, {}^2\mathbf{x}'(s_1^+)\right] \neq 0.$$

If the twice continuously differentiable function $\mathbf{z}(s)$ is the velocity field of an infinitesimal bending of the curve L_A, then the deformed curve $L_{A\varepsilon}$ is determined by the vector equation

$$\mathbf{x} = {}^i\mathbf{x}(s) + \varepsilon\mathbf{z}(s) \qquad (s_{i-1} \leq s \leq s_i;\ i = 1, 2) \tag{2}$$

(ε is the bending parameter, $\mathbf{z}(s)$ is a periodic function with period equal to the length of the curve L_A).

By the condition of the lemma,

$$d\mathbf{x}\, d\mathbf{z} = 0, \tag{3}$$

$$\mathbf{x}''\mathbf{z}'' \geq 0; \tag{4}$$

from (3) it follows that

$$d\mathbf{z} = [\mathbf{y}\, d\mathbf{x}],$$

so that in the case of a trivial infinitesimal bending,

$$\mathbf{z} = \mathbf{a} + [\mathbf{c}\mathbf{x}(s)]$$

($\mathbf{a}, \mathbf{c}$ are constant vectors), while in the case of a nontrivial infinitesimal bending,

$$\mathbf{z}(s) = \mathbf{a} + [\mathbf{c}\mathbf{x}(s)] + \mathbf{z}^*(s). \tag{5}$$

The nontrivial term $\mathbf{z}^*(s)$ satisfies conditions (3) and (4):

$$\mathbf{x}'\mathbf{z}^{*\prime} = 0, \qquad \mathbf{x}''\mathbf{z}^{*\prime\prime} \geq 0,$$

which we shall write in the form

$$\mathbf{t}\mathbf{z}^{*\prime} = 0, \qquad \mathbf{n}\mathbf{z}^{*\prime\prime} \geq 0, \tag{6}$$

where, if by ${}^i\mathbf{t}$ we denote the unit tangent vector to the arc l_i, then for $s = s_i$ we shall simultaneously have

$${}^1\mathbf{t}(s_i) \cdot \mathbf{z}^{*\prime}(s_i) = 0, \qquad {}^2\mathbf{t}(s_i) \cdot \mathbf{z}^{*\prime}(s_i) = 0 \qquad (i = 1, 2).$$

Since, by the condition of the lemma, $[{}^1\mathbf{t}(s_i), {}^2\mathbf{t}(s_i)] \neq 0$, it follows that at the point s_i $(i = 1, 2)$ of the curve L_A the vector $\mathbf{z}^{*\prime}(s)$ is orthogonal to the plane of the curve.

Since

$$\mathbf{n}\mathbf{z}^{*\prime\prime} = (\mathbf{n}\mathbf{z}^{*\prime})' - \mathbf{n}'\mathbf{z}^{*\prime} = (\mathbf{n}\mathbf{z}^{*\prime})' + k\mathbf{t}\mathbf{z}^{*\prime} = (\mathbf{n}\mathbf{z}^{*\prime})',$$

the second of relations (6) can be written in the form

$$d(\mathbf{n}\mathbf{z}^{*\prime}) \geq 0;$$

integrating this total differential along the curve l_i, we obtain

$$\mathbf{n}\mathbf{z}^{*\prime} = 0,$$

whence we conclude that also at the interior points of the arcs l_i the vector $\mathbf{z}^{*\prime}$ is orthogonal to the plane of the curve L_A, i.e.

$$\mathbf{z}^{*\prime} = g(s)\mathbf{b}.$$

($\mathbf{b}$ is a constant unit vector of the binormal, $g(s)$ is a continuous periodic function). Integrating the last equation, we obtain (5) in the form

$$\mathbf{z}(s) = \mathbf{a} + [\mathbf{c}\,\mathbf{x}(s)] + h(s)\mathbf{b};$$

this proves the lemma.

3. We shall prove the theorem formulated above for the case $n = 2$, i.e., for the ribbed cylindroid C_A with two edges σ_1 and σ_2.

Let

$$\mathbf{x}(s,v) = \begin{cases} {}^1\mathbf{x}_1(s) + v\mathbf{a}(s), & s_0 \le s \le s_1, \\ {}^2\mathbf{x}_1(s) + v\mathbf{a}(s), & s_1 \le s \le s_2, \end{cases} \qquad (0 \le v \le 1;\ s_2 \equiv s_0) \tag{7}$$

be the equation of the surface C_A (where the origin of the vector $\mathbf{x}(s,v)$ is in the plane of the curve L_1); the regular functions ${}^i\mathbf{x}_1(s)$ admit regular continuations, and $\mathbf{a}(s)$ is a periodic function with period equal to the length of the curve L_1. Obviously,

$$ {}^1\mathbf{x}_1(s_i) = {}^2\mathbf{x}_1(s_i), \tag{8}$$

and, by the condition of the theorem,

$$[{}^1\mathbf{x}_1'(s_1^-), {}^2\mathbf{x}_1'(s_1^+)] \neq 0, \quad [{}^2\mathbf{x}_1'(s_2), {}^1\mathbf{x}_1'(s_0^+)] \neq 0;$$

moreover

$$\mathbf{a}'(s) = \lambda(s)\, {}^i\mathbf{x}_1'(s),$$

where the continuous periodic scalar function $\lambda(s)$ $(1 + \lambda(s) > 0)$ is such that, for $s = s_i$,

$$\mathbf{a}'(s_i^-) = \lambda(s_i)\, {}^i\mathbf{x}_1'(s_i^-), \quad \mathbf{a}'(s_i^+) = \lambda(s_i)\, {}^i\mathbf{x}_1'(s_i^+).$$

Let $\mathbf{z} = \mathbf{z}(s,v)$, $\mathbf{y} = \mathbf{y}(s,v)$ be, respectively, the velocity field and the rotation field of an infinitesimal bending of the surface C_A. As is known ([2]), on each regular part of the surface C_A, $\mathbf{z}(s,v)$ satisfies the equation

$$d\mathbf{x}\, d\mathbf{z} = 0, \tag{9}$$

whence follows the condition

$$d\mathbf{z} = [{}^i\mathbf{y}, d\, {}^i\mathbf{x}] \qquad (i = 1, 2). \tag{10}$$

The vectors ${}^i\mathbf{y}_s, {}^i\mathbf{y}_v$ lie in the tangent plane of the surface C_A; in the relations

$$\begin{aligned} {}^i\mathbf{y}_s &= {}^i\alpha\, {}^i\mathbf{x}_s - {}^i\beta\, {}^i\mathbf{x}_v, \\ {}^i\mathbf{y}_v &= {}^i\gamma\, {}^i\mathbf{x}_s - {}^i\alpha\, {}^i\mathbf{x}_v \end{aligned} \qquad (i = 1, 2) \tag{11}$$

the scalar differentiable functions ${}^i\alpha(s,v), {}^i\beta(s,v), {}^i\gamma(s,v)$ are determined from the Gauss-Codazzi equations. In the case under consideration, the latter lead to two systems of equations

$$ {}^{i}\alpha_v = -\frac{2\lambda(s)}{1+v\lambda(s)}\,{}^{i}\alpha, \qquad (i=1,2) \tag{12} $$
$$ {}^{i}\alpha_s - {}^{i}\beta_v = 0, $$

whose integration leads to the solutions

$$ {}^{i}\mathbf{y}(s,v) = \int_{s_{i-1}}^{s} \left\{A_i(s)\,{}^{i}\mathbf{x}_1'(s) - B_i'(s)\mathbf{a}(s)\right\}\,ds - \frac{vA_i(s)}{1+v\lambda(s)}\,\mathbf{a}(s) + \mathbf{c}_i, \tag{13} $$

where $A_i(s)$ and $B_i(s)$ are arbitrary scalar functions, and $\mathbf{c}_i$ are constant vectors.

The most general form of the function $\mathbf{y}(s,v)$ corresponds to infinitesimal bendings under which all chords of the curves L_1 and L_2 decrease or remain stationary; in this case, as follows from the lemma proved above, on the boundaries L_1 and L_2 of the ribbed cylindroid C_A the nontrivial component of the velocity field of the infinitesimal bending is orthogonal to the planes of the curves L_1 and L_2. Hence we obtain the boundary conditions for the field $\mathbf{y}(s,v)$

$$ \overset{i}{\beta}(s,0) = \overset{i}{\beta}(s,1) = 0, \tag{14} $$

on the basis of which the functions $A_i(s)$ and $B_i(s)$ in (7) are determined:

$$ A_i(s) = c(1+\lambda(s)), \qquad B_i(s) = 0; $$

the fields of rotations of the regular parts

$$ \overset{i}{\mathbf{y}}(s,v) = c\left\{\overset{i}{\mathbf{x}}_1(s) + \frac{1-v}{1+v\lambda(s)}\,\mathbf{a}(s)\right\} + \mathbf{c}_i \qquad (i=1,2) \tag{15} $$

are nontrivial for all $c \neq 0$.

On the rib σ_i, for example on σ_0 $(s = s_0 = s_2)$, we have

$$ \overset{1}{\mathbf{y}}(s_0,v) = c\left\{\overset{1}{\mathbf{x}}_1(s_0) + \frac{1-v}{1+v\lambda(s)}\,\mathbf{a}(s_0)\right\} + \mathbf{c}_1, $$

$$ \overset{2}{\mathbf{y}}(s_0,v) = c\left\{\overset{2}{\mathbf{x}}_1(s_0) + \frac{1-v}{1+v\lambda(s)}\,\mathbf{a}(s_0)\right\} + \mathbf{c}_2. \tag{16} $$

The velocity field $\mathbf{z}(s,v)$ is single-valued on the whole surface C_A if along σ_i

$$ [\overset{2}{\mathbf{y}}\,d\overset{2}{\mathbf{x}}] = [\overset{1}{\mathbf{y}}\,d\overset{1}{\mathbf{x}}], $$

where $\overset{i}{\mathbf{x}}$ is the vector $\mathbf{x}(s,v)$ describing the corresponding regular part of the surface C_A. Since along σ_i

$$d\overset{2}{\mathbf{x}} = d\overset{1}{\mathbf{x}} = \overset{i}{\mathbf{x}}_v\,dv = \mathbf{a}(s_i)\,dv,$$

the last relation may be written in the form

$$[\overset{2}{\mathbf{y}} - \overset{1}{\mathbf{y}},\, \mathbf{a}(s)] = 0 \qquad (\text{along } \sigma_i)$$

or

$$[\mathbf{y}, \mathbf{a}(s)] = 0 \qquad (\text{along } \sigma_i) \tag{17}$$

($\mathbf{y} = \overset{2}{\mathbf{y}} - \overset{1}{\mathbf{y}}$; thus the vector $\mathbf{y}$ along σ_i is collinear with the vector $\mathbf{a}(s_i)$). On the other hand, taking (1) into account, from (16) we derive:

$$\mathbf{y} = \overset{2}{\mathbf{y}} - \overset{1}{\mathbf{y}} = \mathbf{c}_2 - \mathbf{c}_1 = \mathbf{C}; \tag{18}$$

in the case where the vectors $\mathbf{a}(s_i)$ are not collinear, it follows from (17) and (18) that along σ_i, $\mathbf{y}(s,v) = 0$.

The nontrivial, for $c \neq 0$, field of rotations

$$\mathbf{y}(s,v) = \begin{cases} c\left\{\overset{i}{\mathbf{x}}_1(s) + \dfrac{1-v}{1+v\lambda(s)}\,\mathbf{a}(s)\right\} + \mathbf{C}, & s \neq s_i, \\ \mathbf{C}, & s = s_i \end{cases} \qquad (\mathbf{C} = \text{const} \geq 0).$$

ensures the integrability of the equation $d\mathbf{z} = [\mathbf{y}\,d\mathbf{x}]$.

Applying directly the arguments of N. V. Efimov ([2]), we derive that under the boundary conditions (14) the field $\mathbf{y}$ is single-valued on the surface C_A only in the case when $c = 0$ and $\mathbf{y} = \mathbf{C}$ (≥ 0). The theorem is proved.

4. As is known, the fields $\mathbf{z}$ and $\mathbf{y}$ are invariant under projective transformations, so that the theorem proved is also valid for ribbed cylindroids with nonparallel bases that are projectively transformable into ribbed cylindroids with parallel bases (the boundary conditions being preserved).

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REFERENCES

[1] N. V. Efimov, *Matem. sborn.*, **20**, 1 (1947). [2] N. V. Efimov, *UMN*, **3**, 2 (1948).

Note: Figure translations are in progress. See original paper for figures.