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Abstract

Full Text

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INFINITESIMAL DEFORMATIONS OF SURFACES UNDER SLEEVE CONSTRAINTS

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In the present work infinitesimal deformations of surfaces of positive curvature with a prescribed change of the linear element under certain sleeve constraints are studied.

It is said ([1]) that a surface S with boundary L is subject to a sleeve constraint if L lies on some surface Σ and, in the process of deformation of S, the contour L moves along the surface Σ, sliding over it. Infinitesimal deformations of a surface S with a prescribed change of the linear element under sleeve constraints will henceforth be called simply sliding deformations over Σ.

The first part of the work is devoted to sliding deformations of caps whose boundary is a circle. It is established that a cap S admits one and only one sliding deformation over the surface Σ if the angle $\varphi = \varphi(s)$ between S and Σ along L varies within certain limits and some point of the surface S is fixed. In particular, if the deformations under consideration are infinitesimal bendings (i.e. the change of the linear element of S is identically zero), then under the indicated conditions the surface admits one and only one trivial sliding bending over the surface Σ. The bounds for the variation of the angle φ between S and Σ along L depend only on the angle ψ $(0 < \psi < \pi/2)$ of inclination of the tangent planes of S along L to the base plane; namely,

$$-\pi/2 + \psi(s) \leq \varphi(s) \leq \pi/2.$$

It is shown that for a number of surfaces the left-hand bound of this inequality can be decreased.

In the second part of the work infinitesimal bendings under sleeve constraints are considered for surfaces, star-shaped with respect to some point, whose boundary is an ellipse.

Some sections of the book of I. N. Vekua ([1]), the book of A. V. Pogorelov ([2]), and the work of I. Kh. Sabitov ([3]) are devoted to related questions.

§ 1. We consider simply connected surfaces S of positive curvature $K \geq k_0 > 0$ with boundary L, given with respect to the coordinate system $Oxyz$ by the equation $z = f(x, y)$ and situated convexly downward. We assume that $S \in C^3_\alpha,\ 0 < \alpha < 1;$ L is a circle parallel to Oxy with center on the axis Oz.

Let D denote the disk of radius R in the plane Oxy onto which S is projected, and Γ its boundary. Prescribe in the domain D the quadratic form σ:

$$\sigma(dx, dy) \equiv \sigma_{11}(x,y)\, dx^2 + 2\sigma_{12}(x,y)\, dx\, dy + \sigma_{22}(x,y)\, dy^2,$$

where $\sigma_{ij}(x,y) \in C^2_\alpha(D+\Gamma),\ 0 < \alpha < 1$, and subject the surface S to an infinitesimal deformation with displacement field

$$\vec{\tau}(x,y) = \xi(x,y)\mathbf{i} + \eta(x,y)\mathbf{j} + \zeta(x,y)\mathbf{k}.$$

It is said that the surface S undergoes an infinitesimal deformation with prescribed change of the linear element if

$$dx\, d\xi + dy\, d\eta + df(x,y)\, d\zeta = \sigma(dx, dy). \tag{1}$$

In particular, if $\sigma = 0$, then the corresponding deformation is an infinitesimal bending of the surface S.

Putting

$$\lambda = \xi + \frac{\partial f}{\partial x}\zeta; \qquad \mu = \eta + \frac{\partial f}{\partial y}\zeta;$$
$$w(z) = \lambda(x,y) + i\mu(x,y); \qquad z = x + iy;$$

equation (1) of an infinitesimal deformation of the surface with a prescribed change of the linear element can be written in the form

$$\frac{\partial w}{\partial \bar z} - 2\frac{\partial^2 f}{\partial \bar z^2}\zeta = F;$$
$$\operatorname{Re}\left(\frac{\partial w}{\partial z}\right) - 2\frac{\partial^2 f}{\partial z\,\partial \bar z}\zeta = F_1, \tag{2}$$

where

$$F = \frac{\sigma_{11} - \sigma_{22}}{2} + i\sigma_{12}; \qquad F_1 = \frac{\sigma_{11} + \sigma_{22}}{2};$$
$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right); \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right); \qquad z \in D.$$

We shall say that the point M_0 is fixed under an infinitesimal deformation of the surface S if

$$\xi(M_0) = \eta(M_0) = \zeta(M_0) = \sigma(M_0) = 0. \tag{3}$$

In what follows we assume that the point $(0, 0, f(0,0))$ is fixed.

2. Let S along L be subject to the sleeve constraint Σ. Denote by $\mathbf{n}_\Sigma$ the normal to the surface Σ. Then one may assume that the vector $\mathbf{n}_\Sigma$ has coordinates $(\operatorname{tg}\gamma\, x/R;\ \operatorname{tg}\gamma\, y/R;\ -1)$, where $\gamma = \gamma(s)$ is the angle between $\mathbf{n}_\Sigma$ and the axis Oz. Here $\gamma > 0$ or $\gamma < 0$ according as the projection of the vector $\mathbf{n}_\Sigma$ onto the plane Oxy has the direction of the exterior, or respectively the interior, normal to Γ; $\gamma(s) = 0$ if $\mathbf{n}_\Sigma$ is orthogonal to the plane Oxy. We shall suppose that $\gamma(s) \in C_\alpha(\Gamma)$, $0 < \alpha < 1$.

The condition that the edge L slide along the surface Σ under a deformation of S means that $\mathbf{n}_\Sigma \cdot \bar{\mathbf{t}} = 0$ on Γ, i.e.

$$\operatorname{Re}\{\operatorname{tg}\gamma(t)\cdot \bar{t}w(t)\} - \left(|t| + 2t\ \operatorname{tg}\gamma(t)\frac{\partial f}{\partial t}\right)\zeta(t) = 0; \qquad t \in \Gamma. \tag{4}$$

Thus, the study of deformations of the surface S under sleeve constraints reduces to the study of the solvability of the boundary-value problem (2), (4), where $w(z)$ and $\zeta(z)$ are the unknowns, and $F(z)$, $F_1(z)$, $f(z)$, $\gamma(t)$ are given functions.

3. We shall call problem (2), (4) homogeneous if $F(z) \equiv 0$, $F_1(z) \equiv 0$; otherwise, nonhomogeneous. In addition, to the boundary-value problem (2), (4) we adjoin the requirement (3).

Lemma. *Suppose that on the contour Γ the condition*

$$|t| + 2t\ \operatorname{tg}\gamma(t)\frac{\partial f}{\partial t} > 0. \tag{5}$$

is satisfied. Then the homogeneous boundary-value problem (2), (3), (4) has one and only one solution $w(z) = icz$, $\zeta(z) \equiv 0$, where c is an arbitrary real constant. The nonhomogeneous boundary-value problem (2), (3), (4) has a one-parameter family of solutions for any right-hand sides $F(z)$ and $F_1(z)$. The general solution has the form

$$w(z) = w_0(z) + icz; \qquad \zeta(z) = \zeta_0(z), \tag{6}$$

where $w_0(z)$, $\zeta_0(z)$ is the unique particular solution of the nonhomogeneous problem, with

$$w_0(z) \in C^2_\alpha(D+\Gamma), \qquad \zeta_0(z) \in C^2_\alpha(D+\Gamma), \qquad 0 < \alpha < 1.$$

By virtue of the results of the book ([1]), the assertion of the lemma remains valid if

$$|t| + 2t\tan\gamma(t)\frac{\partial f}{\partial t} = 0, \qquad t \in \Gamma.$$

§4. Let a surface S, satisfying the conditions of §1, be subjected along the edge L to a bushing constraint Σ. Let $\mathbf{n}_S$ be the interior normal of the surface S. We orient the edge L so that, when traversing the contour L, the surface S lies on the left. Choose on the surface Σ, at some point M, that direction of the normal $\mathbf{n}_\Sigma$ which makes an acute angle with $\mathbf{n}_S$. At the remaining points of the edge, continue the direction of the normal to the surface Σ by continuity. At each point of the edge the vectors $\mathbf{n}_S$ and $\mathbf{n}_\Sigma$ lie in the plane Ozr, passing through the axis Oz. The equation of the line of intersection of the surface S with the plane Ozr in the coordinate system Ozr may be written in the form $z = f(r)$, where the axis Or has the direction of the exterior normal Γ. Denote by φ the angle between $\mathbf{n}_\Sigma$ and $\mathbf{n}_S$. We measure the angle from $\mathbf{n}_S$ to $\mathbf{n}_\Sigma$ counterclockwise, if viewed from the side of the direction of the curve L. Obviously, the angle φ is a function of the arc length of the curve L, i.e. $\varphi = \varphi(s)$.

From the lemma there follows the following theorem.

Theorem 1. *A surface S, fixed at the point $(0, 0, f(0, 0))$, and subjected to a bushing constraint Σ, satisfying the condition*

$$-\frac{\pi}{2} + \psi(s) \leqslant \varphi(s) \leqslant \frac{\pi}{2}, \tag{7}$$

where $\psi(s) = \arctan \dfrac{df(r)}{dr}$, admits an infinitesimal sliding deformation along Σ with a prescribed change of the linear element of the surface S, depending on one parameter. This deformation is composed of a rotation of the surface S as a rigid body about the axis Oz (this is caused by the solution of the homogeneous problem in (6)) and a deformation of the surface with the fixed point together with its tangent plane (this is caused by a particular solution of the nonhomogeneous problem in (6)).

In the case of infinitesimal bendings, the surface S under bushing constraints satisfying condition (7) admits one and only one trivial infinitesimal bending (a rotation about the axis Oz).

In some cases, for infinitesimal bendings, Theorem 1 admits a strengthening. Namely, the following holds.

Theorem 2. *Let S be a cap of a surface of revolution, bounded by a parallel of radius R and turned with its convexity downward. Let $z = f(r)$ be the equation of the meridian in the plane Ozr. Suppose, furthermore, that on S the following conditions are satisfied:*

$$f'_r f''_{rr} + r(f'_r f'''_{rrr} - f''^{\,2}_{rr}) \geqslant 0 \quad \text{in the domain } D,$$

$$f'_r - R f''_{rr} > 0 \quad \text{on the contour } \Gamma.$$

Then the surface S, fixed at some point, is rigid if the bushing constraint is subject to the condition:

$$-\frac{\pi}{2} + [\psi(s) - \psi_1(s)] \leqslant \varphi(s) \leqslant \frac{\pi}{2},$$

where $\psi_1(s) = \arctan(f'_r - R f''_{rr})$.

For spherical caps, in particular, the following holds.

Theorem 3. *Let S be a spherical cap of the unit sphere, fixed at its vertex. If the bushing constraint of the cap S satisfies the condition*

$$-\arctan\frac{1-h}{h(2-h)} \leqslant \varphi(s) \leqslant \frac{\pi}{2},$$

where h is the height of the cap, then S is rigid.

Analogous propositions hold also for the class of surfaces that are not surfaces of revolution.

§5. Application of the Darboux–Sauer theorem (see, for example, ([2])) on the connection between infinitesimal bendings and projective transformations of space makes it possible to prove a number of propositions on the rigidity of surfaces, star-shaped

with respect to a certain point and subject to sleeve constraints. Thus, from Theorem 1 we obtain Theorem 4.

Theorem 4. *Let S be a convex surface, star-shaped with respect to a center O, separated from the surface by some plane, and let the boundary of the surface be an ellipse. Suppose further that S is subject to a sleeve constraint Σ satisfying the condition*

$$-\frac{\pi}{2} + \psi(s) \leqslant \varphi(s) \leqslant \frac{\pi}{2},$$

where $\varphi(s)$ and $\psi(s)$ have the same meaning as above. Then, if some point is fixed, the surface S is rigid.

6. Let now S be a surface that projects one-to-one onto the plane Oxy as a convex domain. Suppose that along L a continuous variable vector field $\vec{n}_\varepsilon$ is prescribed, each vector of which lies in the plane passing through the axis Oz and the boundary point. The surface S is said to admit infinitesimal deformations of generalized sliding if the vector $\vec{\tau}$ of the deformation field at each boundary point satisfies the condition: $n_z \cdot \vec{\tau} = 0$. In the study of deformations of generalized sliding of the surface S, one can prove a statement analogous to the lemma given above. However, the condition for solvability of the problem has a more cumbersome form. It depends on the first and second partial derivatives of the surface $z = f(x, y)$, on the equation of the contour L, and on its first derivative. This condition makes it possible to prove theorems analogous to those given above for infinitesimal deformations of generalized sliding, in particular for infinitesimal bendings of generalized sliding.

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REFERENCES

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3. I. Kh. Sabitov, DAN, 147, No. 4 (1962).

Note: Figure translations are in progress. See original paper for figures.