



Soviet-era science, translated into English

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1964

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Abstract

Full Text

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ON THE REDUCTION PRINCIPLE IN THE THEORY OF STABILITY OF MOTION

(Presented by Academician V. I. Smirnov, 14 X 1963)

The question of reduction in the theory of stability consists in the following. Consider a system of differential equations of the form

$$\frac{dx}{dt} = Ax + p(x, y), \quad \frac{dy}{dt} = By + q(x, y), \quad (1)$$

where x is an n -dimensional vector; y is an m -dimensional vector; A and B are square matrices of orders n and m , respectively; p and q are vector functions whose norms $\|p\|$ and $\|q\|$ are small in comparison with the norms of the vectors x and y . It is assumed that all eigenvalues of the matrix A have zero real parts. The question is posed of the existence and construction of such an n -dimensional vector function $p^*(x)$ that the problem of stability of the zero solution of the system

$$\frac{dx}{dt} = Ax + p^*(x) \quad (2)$$

would be equivalent to the problem of stability of the zero solution of system (1).

The question of the reduction principle was studied by I. G. Malkin ^(1,2), V. N. Postnikov ⁽³⁾, K. P. Persidskii ⁽⁴⁾, E. I. Dykhman ⁽⁵⁾, and S. Lefschetz ⁽⁶⁾. Mainly the analytic case was considered. It was proved (see, for example, ⁽²⁾, pp. 373-374) that if the zero solution of the system

$$\frac{dx}{dt} = Ax + p(x, 0) \quad (3)$$

is stable, asymptotically stable, or unstable independently of terms of order higher than some fixed natural number N , and if the expansions of the components of the vector function $q(x, 0)$ begin with terms of order not lower than $N + 1$, then the zero solution of system (1) is respectively stable, asymptotically stable, or unstable. A transformation was also found which in some cases allows one to bring the original system to such a form that the stated reduction principle is valid for it.

The reduction principle established in the works mentioned is valid only in algebraic cases ⁽⁷⁾, for only in these cases can one speak of stability independently of terms of order higher than N . For transcendental cases, however, even the question of the very possibility of reduction remains open.

In the present note the existence is established of such an m -dimensional vector function $f(x)$ that, if the zero solution of the system

$$\frac{dx}{dt} = Ax + p(x, f(x)) \quad (3')$$

is stable, asymptotically stable, or unstable, then the zero solution of system (1) is respectively stable, asymptotically stable, or unstable. Thus the validity of the reduction principle is established both in algebraic and in transcendental cases.

1. In all that follows we shall assume that the functions p and q are defined in a neighborhood of the origin and satisfy in this neighborhood

the Lipschitz condition with an infinitely small constant:

$$\begin{aligned} \|p(x_1, y_1) - p(x_2, y_2)\| &< \alpha(\|x_1 - x_2\| + \|y_1 - y_2\|), \\ \|q(x_1, y_1) - q(x_2, y_2)\| &< \alpha(\|x_1 - x_2\| + \|y_1 - y_2\|), \end{aligned} \quad (4)$$

where $\alpha \rightarrow 0$ as $\|x_1\| + \|x_2\| + \|y_1\| + \|y_2\| \rightarrow 0$.

Moreover, we assume, of course, that

$$p(0, 0) = 0, \quad q(0, 0) = 0. \quad (5)$$

By $x(t - t_0, x_0, y_0)$, $y(t - t_0, x_0, y_0)$ we shall denote the solution of system (1) with initial data $t = t_0$, $x = x_0$, $y = y_0$.

Using the ideas expressed by N. N. Bogolyubov in the monographs ^(8, 9), one can prove the following assertion.

Theorem 1. *If the eigenvalues of the matrix A have nonnegative real parts, while the eigenvalues of the matrix B have negative real parts, and if conditions (4) and (5) are satisfied, then there exists an m -dimensional vector function $f(x)$ with the following properties:*

- 1) $f(x)$ is defined and satisfies the Lipschitz condition with unit constant:

$$\|f(x_1) - f(x_2)\| \leq \|x_1 - x_2\|$$

for $\|x\| \leq h$, where $h > 0$ is sufficiently small;

- 2) $\frac{\|f(x)\|}{\|x\|} \rightarrow 0$ as $\|x\| \rightarrow 0$;
- 3) if $\|x_0\| < h$, $y_0 = f(x_0)$, then

$$y(t, x_0, y_0) = f(x(t, x_0, y_0))$$

as long as $\|x(t, x_0, y_0)\| \leq h$.

The proof of this theorem is carried out on the basis of Banach's principle; therefore the function $f(x)$ can be constructed by the method of successive approximations.

From the third assertion of the theorem just formulated it follows that the surface

$$y = f(x) \tag{6}$$

is invariant.

2. We now formulate the reduction principle.

Theorem 2. *Let the eigenvalues of the matrix A have zero real parts, and let the eigenvalues of the matrix B have negative real parts; let conditions (4) and (5) be satisfied. Then, if the zero solution of the system*

$$\frac{dx}{dt} = Ax + p(x, f(x)), \tag{7}$$

where $f(x)$ is the function furnished by Theorem 1, is stable, asymptotically stable, or unstable, then the zero solution of system (1) is, respectively, stable, asymptotically stable, or unstable.

This theorem is especially simple to prove if instability of the zero solution of system (7) occurs. Indeed, since the surface (6) is invariant, the behavior of solutions on it is described by system (7), and therefore, if the zero solution of this system is unstable, then the equilibrium state $x = 0$, $y = 0$ on the invariant surface (6) is also unstable. Hence follows the instability of the zero solution of system (1).

In the case where the zero solution of system (7) is stable, the proof of Theorem 2 rests on the following considerations. Consider the solution of system (1) that starts on the invariant surface (6): $x = x(t, \xi, f(\xi))$, $y = y(t, \xi, f(\xi))$. Since the zero solution of system (7) is stable, for all $t \geq 0$ $\|x\|$ and $\|y\|$ are arbitrarily small, provided only that $\|\xi\|$ is sufficiently—

sufficiently small. Having noted this, we make the change of variables:

$$x = x(t, \xi, f(\xi)) + \varphi, \quad y = y(t, \xi, f(\xi)) + \psi. \tag{8}$$

The system of equations for φ and ψ has the form

$$\frac{d\varphi}{dt} = A\varphi + p_1(\varphi, \psi, \xi, t), \quad \frac{d\psi}{dt} = B\psi + q_1(\varphi, \psi, \xi, t). \quad (9)$$

It is not difficult to see that

$$p_1(0, 0, \xi, t) = 0, \quad q_1(0, 0, \xi, t) = 0 \quad (10)$$

and the functions p_1 and q_1 satisfy the Lipschitz condition:

$$\begin{aligned} \|p_1(\varphi_1, \psi_1, \xi, t) - p_1(\varphi_2, \psi_2, \xi, t)\| &< \beta(\|\varphi_1 - \varphi_2\| + \|\psi_1 - \psi_2\|), \\ \|q_1(\varphi_1, \psi_1, \xi, t) - q_1(\varphi_2, \psi_2, \xi, t)\| &< \beta(\|\varphi_1 - \varphi_2\| + \|\psi_1 - \psi_2\|), \end{aligned} \quad (11)$$

where the constant β can be made arbitrarily small by choosing sufficiently small $\|\xi\|$, $\|\varphi_1\|$, $\|\varphi_2\|$, $\|\psi_1\|$, $\|\psi_2\|$.

Using the estimates (11), by the method of successive approximations one can prove that, for each ξ , the system (9) has an m -parameter family of solutions tending to zero:

$$\varphi = \varphi(t, \xi, a), \quad \psi = \psi(t, \xi, a), \quad (12)$$

where a is an m -dimensional vector. Moreover, the functions $\varphi(t, \xi, a)$ and $\psi(t, \xi, a)$ are defined and continuous for $t \geq 0$, $\|a\| \leq a_0$, $\|\xi\| \leq \xi_0$, where $a_0 > 0$, $\xi_0 > 0$ are sufficiently small, and satisfy the estimates

$$\begin{aligned} \|\varphi(t, \xi, a)\| &\leq C\beta\|a\|e^{-\lambda t}, \\ \|\psi(t, \xi, a)\| &\leq C\|a\|e^{-\lambda t}, \end{aligned} \quad (13)$$

where C and λ are certain positive constants.

Take arbitrary vectors x_0 and y_0 with sufficiently small norms. It can be proved that there exist such ξ and a , with sufficiently small norms, that

$$x_0 = \xi + \varphi(0, \xi, a), \quad y_0 = f(\xi) + \psi(0, \xi, a).$$

Consider the solution $x = x(t, x_0, y_0)$, $y = y(t, x_0, y_0)$ of system (1). The change (8) shows that for all $t \geq 0$ the following equalities will hold:

$$x(t, x_0, y_0) = x(t, \xi, f(\xi)) + \varphi(t, \xi, a),$$

$$y(t, x_0, y_0) = y(t, \xi, f(\xi)) + \psi(t, \xi, a). \quad (14)$$

From these equalities, the stability of the zero solution of system (7), and the estimates (13), it follows that $\|x(t, x_0, y_0)\|$ and $\|y(t, x_0, y_0)\|$ for $t \geq 0$ are arbitrarily small if $\|\xi\|$ and $\|a\|$ are sufficiently small; hence the stability of the zero solution of system (1) follows.

From the equalities (14) and the estimates (13) it also follows that the zero solution of system (1) is asymptotically stable if the zero solution of system (7) is asymptotically stable.

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Received
9 X 1963

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