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Abstract

Full Text

MATHEMATICS

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ON UNIFORM AND ASYMPTOTIC APPROXIMATION BY ENTIRE FUNCTIONS ON UNBOUNDED CLOSED SETS

(Presented by Academician M. V. Keldysh on 8 II 1964)

One of the fundamental problems in the theory of uniform approximation by entire functions is the following: on which closed sets E does every function continuous on E (except, possibly, at the point $z = \infty$) and analytic at the interior points of E admit uniform approximation by entire functions? As Carleman first established ⁽¹⁾, for E one may take any Jordan curve beginning and ending at infinity. This result was substantially strengthened by M. V. Keldysh and M. A. Lavrent'ev ⁽²⁾, who established a necessary and sufficient condition on a nowhere dense continuum E in order that every function continuous on E should admit uniform approximation by entire functions. This condition is the following.

Condition K. We shall say that a closed set E satisfies condition K, or is a K -set, if there exists a function $r_E(t) \geq 0$ increasing to $+\infty$ such that every point $z \in CE$ can be joined with infinity by a Jordan line lying in CE and outside the circle $|\zeta| \leq r_E(|z|)$.

For sets E containing interior points, M. V. Keldysh established ^(3,4) that condition K for continua is sufficient for the possibility of approximation. It turns out that condition K in the general case is not only sufficient but also necessary for the solution of the problem formulated at the beginning of this note.

Theorem 1. *In order that every function continuous on the closed set E (except, perhaps, at the point $z = \infty$) and analytic at the interior points of E admit uniform approximation on E by entire functions, it is necessary and sufficient that E satisfy condition K or coincide with the whole plane.*

The proof of the necessity of condition K is based on the following lemma.

Lemma 1. *Let γ be a Jordan arc lying in the annulus*

$$S: \quad r \leq |z| \leq R$$

and joining the boundary circles of S . For the function $\varphi(z)$, $\varphi(0) = 0$, mapping the exterior of γ onto the unit disk, the estimate holds

$$1 - |\varphi(z)| < \left(\frac{8r}{R-r} \right)^{1/2}, \quad |z| > R.$$

If the set E does not satisfy condition K, then one constructs a function $f(z)$, analytic on E , for which the inequality

$$|f(z) - G(z)| < 1, \quad z \in E, \quad (1)$$

is not satisfied for any entire function $G(z)$. In the case when the complement of E contains a bounded component G lying in the circle $|z - z_0| < d$, $z_0 \in G$, we choose $f(z) = \frac{2d}{z - z_0}$. If, however, such a component G does not exist, then first we choose a certain sequence

Jordan arcs $\{\gamma_k\}_1^\infty \in CE$, $\min_{z \in \gamma_k} |z| \rightarrow \infty$, and a sequence of points $\{a_k\}_1^\infty \in CE$, $a_k \rightarrow \infty$. Next we form the infinite product

$$\prod_{k=1}^{\infty} \frac{f_k(a_k) - f_k(z)}{1 - \overline{f_k(a_k)} f_k(z)} \frac{\overline{f_k(a_k)}}{|f_k(a_k)|},$$

where $f_k(z)$ is the function mapping the exterior of γ_k onto the unit disk, with $f_k(0) = 0$.

With the aid of Lemma 1 it is proved that this product converges uniformly outside the arcs $\{\gamma_k\}_1^\infty$ to a certain analytic function $\mathfrak{B}(z)$, all of whose zeros lie at the points $\{a_k\}_1^\infty$. Finally, it is proved that for the function $f(z) = \frac{2}{\mathfrak{B}(z)}$ there exists no entire function $G(z)$ satisfying inequality (1).

The sufficiency of condition K for Theorem 1 is proved according to a scheme which is a synthesis of certain constructive schemes of M. V. Keldysh and S. N. Mergelyan, developed by them in paper (4). In doing so we use the following lemmas.

Lemma 2. Let σ be a Jordan domain lying in the annulus

$$S: r \leq |z| \leq R,$$

and joining the boundary circles of S . If $20r \leq R$, there exists a rational function $R(z)$ with a pole in σ , satisfying the estimates

$$\left| \frac{1}{z} - R(z) \right| < c \frac{R^2}{|z|^3} \quad \text{for } |z| \geq R,$$

$$|R(z)| < \frac{c}{r} \quad \text{everywhere outside } \sigma,$$

where c is an absolute constant.

This lemma, similar to an important lemma of S. N. Mergelyan ⁽⁴⁾ on approximation of the Cauchy kernel, is easily proved with the aid of Lemma 1.

Lemma 3. Let E be a closed set satisfying condition K with the aid of the function $r_E(t)$. If $R_\zeta(z)$ is a rational function with a single pole at the point $\zeta \in CE$, then for any $\varepsilon > 0$ there exists an entire function $G_\zeta(z)$ such that

$$|R_\zeta(z) - G_\zeta(z)| < \varepsilon$$

on the set E and in the disk $|z| \leq r_E(|\zeta|)$.

Theorem 1 can be applied to the question of asymptotic approximation by entire functions with contact at infinity. A closed set E is said to be a **Carleman set** if every function continuous on E (possibly excluding the point $z = \infty$) admits an approximation of the form

$$|f(z) - G(z)| < \varepsilon(|z|), \quad z \in E, \quad (2)$$

where $\varepsilon(t)$ is any function with positive lower bound on every finite interval.

In the case where E is a continuum, the problem of describing Carleman sets was completely solved in paper ⁽²⁾ by M. V. Keldysh and M. A. Lavrent' ev. The following theorem extends their result to arbitrary closed sets.

Theorem 2. In order that a closed set E be a Carleman set, it is necessary and sufficient that E be nowhere dense and satisfy condition K.

The necessity follows from Theorem 1. To prove sufficiency, take an arbitrary function $\varepsilon(t)$ with positive lower bound

on any finite interval and denote

$$f_1(z) = \ln \left\{ \int_{|z|}^{|z|+1} \left[\inf_{t \leq r} \varepsilon(t) \right] dr \right\}. \quad (3)$$

The function $f_1(z)$ is continuous at every point z ; therefore, by Theorem 1, there exists an entire function $G_1(z)$ such that

$$|f_1(z) - G_1(z)| < 1, \quad z \in E. \quad (4)$$

The entire function $G_2(z) = e^{G_1(z)-1}$ has no zeros in the whole plane and, according to (3) and (4), satisfies the inequality

$$|G_2(z)| < \varepsilon(|z|), \quad z \in E. \quad (5)$$

Now take an arbitrary function $f(z)$ continuous on E . The function $f(z)/G_2(z)$ is also continuous on E , so that, by Theorem 1, there exists an entire function $G_3(z)$ such that

$$\left| \frac{f(z)}{G_2(z)} - G_3(z) \right| < 1, \quad z \in E. \quad (6)$$

The required entire function satisfying (2) is obtained from (5) and (6) by setting $G(z) = G_2(z)G_3(z)$.

In the case of asymptotic approximation of the form (2) on sets containing interior points, the deviation $\varepsilon(t)$ cannot be an arbitrarily rapidly decreasing function. M. V. Keldysh established^(3,4) the possible orders of decrease of $\varepsilon(t)$ depending on the metric properties of the set E . These orders were refined in⁽⁵⁾. Theorem 1 makes it possible to formulate the results in a more complete form. The following theorem is a strengthening of Theorem 1 (it is assumed that $\varepsilon(t)$ is a nonincreasing function).

Theorem 3. *In order that every function $f(z)$, continuous on a closed set E (except, possibly, for the point $z = \infty$) and analytic at the interior points of E , admit an approximation of the form (2) with the condition*

$$\int_1^\infty \frac{\ln \varepsilon(t)}{t^{3/2}} dt > -\infty,$$

it is necessary and sufficient that E be a K -set or coincide with the whole plane. Moreover, if

$$\int_1^\infty \frac{\ln \varepsilon(t)}{t^{3/2}} dt = -\infty,$$

then there exist K -sets E and functions $f(z)$, analytic on E , such that (2) does not hold for any entire function $G(z)$.

The necessity of condition K follows from Theorem 1; the sufficiency can be proved, as in⁽⁵⁾, by applying Theorem 1.

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