Reports of the Academy of Sciences of the USSR

MATHEMATICS

1964

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1964. Volume 158, No. 2

MATHEMATICS

V. V. GRUSHIN

ON THE BEHAVIOR OF SOLUTIONS OF DIFFERENTIAL EQUATIONS NEAR THE BOUNDARY

(Presented by Academician I. G. Petrovskii, 7 IV 1964)

In the present article we clarify conditions under which the solution $u(x_1, \dots, x_n)$ of a certain differential equation, defined in the half-space $x_n > 0$, has a limit as $x_n \to +0$. Here only equations that can be solved with respect to the highest derivative in x_n are considered, and the limit is understood in the sense of convergence of generalized functions (see ([1,9])).

We shall denote by x a point of n-dimensional space, $x = (x_1, \dots, x_n)$, and by x' a point of $(n-1)$-dimensional space, $x' = (x_1, \dots, x_{n-1})$. Every point x can be represented in the form $x = (x', x_n)$. Let a positive number $a > 0$ be given. By $I_n(a)$ we shall denote the open cube of n-dimensional space

$$|x_1| < a, \dots, |x_n| < a,$$

by $I_n^+(a)$ the open parallelepiped

$$|x_1| < a, \dots, |x_{n-1}| < a, \quad 0 < x_n < a,$$

and by $I_{n-1}(a)$ the open cube of $(n-1)$-dimensional space

$$|x_1| < a, \dots, |x_{n-1}| < a.$$

Definition. Let $u(x)$ be a certain generalized function defined in the domain $I_n^+(a)$. We shall say that $u(x)$ **has a limit as** $x_n \to +0$ if, for $0 \le t < a$, in $I_{n-1}(a)$ one can find a generalized function $v_t(x')$, depending continuously on t for all $0 \le t < a$, such that for every basic function $\varphi(x)$ with support in $I_n^+(a)$

$$(u, \varphi) = \int_0^\infty [v_{x_n}(x'), \varphi(x', x_n)]\, dx_n. \tag{1}$$

It is easy to see that there can be only one such function $v_t(x')$. Moreover, the following is readily established.

Lemma 1. *If the generalized function $v_t(x')$, defined in $I_{n-1}(a)$ for all $0 \leq t < a$, depends continuously on the parameter t, then expression (1) defines a certain generalized function in $I_n(a)$.*

Thus, in order that the generalized function $u(x)$, defined in $I_n^+(a)$, should have a limit as $x_n \to +0$, it is necessary (but by no means sufficient) that $u(x)$ can be extended to some generalized function defined in $I_n(a)$.

We now introduce new spaces $G_+^r(a)$. By $G_+^0(a)$ we shall denote the set of all generalized functions defined in $I_n^+(a)$ which have, as $x_n \to +0$, a limit in the sense of the definition given above. If $r > 0$ is an integer, then by $G_+^r(a)$ we shall denote the set of all generalized functions $u(x)$, defined in $I_n^+(a)$, such that

$$u \in G_+^0(a), \quad \partial u/\partial x_n \in G_+^0(a), \dots, \quad \partial^r u/\partial x_n^r \in G_+^0(a).$$

If $r < 0$ and r is an integer, then by G_+^r we shall denote the set of those generalized functions $u(x)$, defined in $I_n^+(a)$, which can be represented in the form of a sum

$$u = \sum_{0 \leq j \leq |r|} \frac{\partial^j u_j}{\partial x_n^j},$$

where all u_j belong to $G_+^0(a)$. By $G_+^\infty(a)$ we shall denote the intersection of all $G_+^r(a)$, and by $G_+^{-\infty}(a)$ their union.

As an example we note that every function absolutely summable in $I_n^+(a)$ belongs to the space $G_+^{-1}(a)$. Directly from the definition of the spaces $G_+^r(a)$ the following is derived.

Lemma 2. *Differentiation $\partial/\partial x_1, \dots, \partial/\partial x_{n-1}$ and multiplication by infinitely differentiable functions map the spaces $G_+^r(a)$ into themselves.*

Somewhat more difficult to establish is

Lemma 3. *If $\partial u/\partial x_n \in G_+^{r-1}(a)$, then $u \in G_+^r(a)$.*

Proof. Let first $r = 1$. From the fact that $\partial u/\partial x_n \in G_+^0(a)$ it follows that for every test function $\varphi(x)$

$$\left(\frac{du}{dx_n}, \varphi\right) = \int_0^\infty \left[w_{x_n}(x'), \varphi(x', x_n)\right] dx_n,$$

where $w_t(x')$ is a generalized function defined in $I_{n-1}(a)$ for all $0 \leq t < a$ and depending continuously on t. To show that $u \in G_+^1(a)$, consider the generalized function $v(x)$, which is defined by the formula

$$(v, \varphi) = \int_0^\infty \int_{a/2}^{x_n} [w_t(x'), \varphi(x', t)]\, dt\, dx_n. \tag{2}$$

From this formula it is clear that $v \in G_+^1(a)$ and $\dfrac{\partial}{\partial x_n}(u - v) = 0$. But it is well known that every generalized function whose derivative with respect to x_n is zero can be represented in the form of the direct product of a generalized function depending only on x' and the unit (see (9), p. 112). Thus it is clear that $u - v \in G_+^{\infty}(a)$, while expression (2) shows that $v \in G_+^1(a)$, whence we obtain $u \in G_+^1(a)$. In the case of arbitrary r the assertion of the lemma is proved by analogous arguments.

The spaces $G_+^r(a)$, which were defined above, are very close to the spaces introduced in the work of T. Kasyug (6). The only difference is that in (6) generalized functions are considered without any conditions on their behavior as $x_n \to 0$. However, it is precisely the behavior as $x_n \to 0$ that will interest us most of all.

Let us now have a system of differential equations

$$\frac{\partial u_i(x)}{\partial x_n} = \sum_{j=0}^{m} \sum_{0 \le |\alpha| \le l} a_{i,j,\alpha}(x) \frac{\partial^{|\alpha|} u_j(x)}{\partial x_1^{\alpha_1} \cdots \partial x_{n-1}^{\alpha_{n-1}}} + b_i(x), \tag{3}$$

where $a_{i,j,\alpha}(x)$ are infinitely differentiable functions. Just as in the work of T. Kasyug (6), one can easily establish the following assertion:

Theorem 1. *Let the generalized functions $u_i(x)$ $(i = 1, \dots, m)$ belong to the space $G_+^{-\infty}(a)$. If $b_i(x) \in G_+^{r-1}(a)$ $(i = 1, \dots, m)$, then $u_i(x) \in G_+^r(a)$.*

Proof. Since $u_i(x) \in G_+^{-\infty}(a)$ $(i = 1, \dots, m)$, for some $-\infty < p < +\infty$ all $u_i(x)$ $(i = 1, \dots, m)$ belong to $G_+^p(a)$. If $p \le r-1$, then the right-hand sides in equality (3) belong to $G_+^p(a)$ and, consequently, $\partial u_i / \partial x_n \in G_+^p(a)$. As Lemma 3 asserts, it follows that $u_i(x) \in G_+^{p+1}(a)$. If $p + 1 \le r - 1$, then this argument can be repeated once more. After a finite number of steps we obtain that $u_i(x) \in G_+^r(a)$. In the particular case when all $b_i(x)$ are equal to zero or belong to $G_+^{\infty}(a)$, we obtain that $u_i(x) \in G_+^{\infty}(a)$.

Theorem 2. *Let the generalized functions $u_i(x)$, defined in $I_n^+(a)$, satisfy the system (3) and let all $b_i(x)$ belong to $G_+^{-1}(a)$. In order that*

$u_i(x)$ have a limit as $x_n \to +0$, it is necessary and sufficient that they extend to generalized functions defined in $I_n(a)$.

Proof. If all $u_i(a)$ extend to generalized functions defined in $I_n(a)$, then in any $I_n(b)$ $(0 < b < a)$ they have finite order and, consequently, belong to $G_+^{-\infty}(b)$. As now follows from Theorem 1, $u_i(x) \in G_+^0(b)$ for any $0 < b < a$. But from this it easily follows that $u_i(x) \in G_+^0(a)$.

We now consider the differential operator

$$P(D)u = a_0 \frac{\partial^m u}{\partial x_n^m} + \sum_{i=0}^{m-1} \sum_{0 \le |\alpha| \le l} a_{i,\alpha} \frac{\partial^{|\alpha|+i} u}{\partial x_1^{\alpha_1} \dots \partial x_{n-1}^{\alpha_{n-1}} \partial x_n^i}, \tag{4}$$

where $a_0 \neq 0$. We shall apply the result obtained above to the solutions of the differential equation $P(D)u(x) = f(x)$. Denoting $u_1 = u$, $u_2 = du/dx_n, \dots, u_m = \partial^{m-1}u/\partial x_n^{m-1}$, we obtain that the functions $u_i(x)$ satisfy a system of the form (3). Therefore Theorems 1 and 2 will be valid for the solutions of such an equation. Applying, in particular, these theorems to a fundamental solution $E(x)$, i.e., to a solution of the equation $P(D)E(x) = \delta(x)$, we obtain the following assertion (cf. the result from [11]):

Theorem 3. If $E(x)$ is an arbitrary fundamental solution of the differential operator (4), then one can find a generalized function $e_t^+(x')$, depending on t for $0 \leq t < \infty$ as a parameter in an infinitely differentiable manner, and a generalized function $e_t^-(x')$, depending on the parameter t for $-\infty < t \leq 0$ in an infinitely differentiable manner, such that

$$(E, \varphi) = \int_{-\infty}^{0} [e_{x_n}^-(x'), \varphi(x', x_n)]\, dx_n + \int_0^{\infty} [e_{x_n}^+(x'), \varphi(x', x_n)]\, dx_n. \tag{5}$$

The generalized functions e_t^+ and e_t^- satisfy, for $t = 0$, the relations

$$e_{+0}^+ = e_{-0}^-, \qquad \frac{de_{+0}^+}{dt} = \frac{de_{-0}^-}{dt}, \dots, \qquad \frac{d^{m-2}e_{+0}^+}{dt^{m-2}} = \frac{d^{m-2}e_{-0}^-}{dt^{m-2}},$$

$$\frac{d^{m-1}e_{+0}^+}{dt^{m-1}} - \frac{d^{m-1}e_{-0}^-}{dt^{m-1}} = \frac{1}{a_0}\delta(x'). \tag{6}$$

The functions $e_{x_n}^+(x')$ for $x_n > 0$ and $e_{x_n}^-(x')$ for $x_n < 0$ are solutions of the differential operator (4). Conversely, if there are e_t^+ and e_t^- with these properties, then expression (5) defines a fundamental solution.

Proof. After everything said above, only the matching conditions (6) need to be checked. To verify these conditions, it is enough to compute $P(D)E(x)$, where $E(x)$ is given by (5), and to set the result obtained equal to $\delta(x)$. Technically, everything reduces to repeated integration by parts of expression (5) with $\varphi(x) = P(D)\psi(x)$ and to computing the nonintegral terms.

Finally, let us apply Theorem 2 to the solutions of a hypoelliptic equation (see [4,10])

$$P(D)u(x) = 0. \tag{7}$$

Theorem 4. In order that a solution $u(x)$ of the hypoelliptic equation (7), defined in the domain $I_n^+(a)$, have a limit as $x_n \to +0$, it is necessary and sufficient that, for every $0 < b < a$, in the domain $I_n^+(b)$ the inequality

$$|u(x)| \leq C(b)/x_n^{r(b)}$$

hold with some constants $C(b) > 0$ and $r(b) > 0$, depending on b.

This follows directly from Theorem 2 and the following assertion:

Theorem 5. Let, in some domain Ω, there be a solution $u(x)$ of the hypoelliptic equation (7). In order that $u(x)$ can be extended as a generalized function to the whole space, it is necessary and sufficient that for every $b > 0$ there exist constants $C(b) > 0$ and $r(b) > 0$ such that in the domain $\Omega \cap I_n(b)$

$$|u(x)| \le C(b)/[\rho(x, \dot{\Omega})]^{r(b)}.$$

In turn, this theorem is a consequence of the following assertion:

Lemma 4. Let $P(D)$ be a hypoelliptic operator. For any $m > 0$ one can find constants C and k such that for all $0 < b < 1$

$$\left\| u(x),\, I_n\left(\frac{b}{2}\right) \right\|_m \le \frac{C}{b^k} \left\| u(x),\, I_n(b) \right\|_{-m}, \tag{8}$$

where $u(x)$ is an arbitrary solution of equation (7) in the domain $I_n(b)$, and $\|u(x), \Omega\|_m$ is the norm in the corresponding space $W_2^m(\Omega)$.

Proof. Let $\varphi(x)$ be an infinitely differentiable function equal to one for $x \in I_n(b/8)$ and equal to zero for $x \notin I_n(b/4)$. Applying to $u(x)$ the mean-value theorem (see ([10])), we obtain that, for $x \in I_n(b/2)$,

$$u(x) = \int u(\xi)\Psi(x - \xi)\, d\xi, \tag{9}$$

where $\Psi(x) = P(D)[1 - \varphi(x)]E(x)$, and $E(x)$ is a fundamental solution. Inequality (8) easily follows from relation (9), if one takes into account that the fundamental solution $E(x)$ and its derivatives as $x \to 0$ have power growth (see ([2])).

Remark. Theorems 4 and 5 are also valid for solutions of formally hypoelliptic equations with variable coefficients (see ([4,5,7,8])). The fact that the operator $P(D)$ has constant coefficients was used by us only in the proof of Lemma 4. An assertion analogous to this lemma also holds for operators with variable coefficients.

Lemma 5. Let $P(x, D)$ be a formally hypoelliptic operator. For any number $m > 0$ and compact set K one can find constants C and k such that for all $0 < b < 1$

$$\left\| u(x+y),\, I_n\left(\frac{b}{2}\right) \right\|_m \le \frac{C}{b^k} \left\| u(x+y),\, I_n(b) \right\|_{-m}, \tag{10}$$

where $u(x)$ is an arbitrary solution of the equation $P(x, D)u(x) = 0$ in the domain $K + I_n(b)$, and y is an arbitrary point of K.

To prove this assertion, instead of the fundamental solution one may now use the "parametrix"constructed in ([4,8]), first estimating the singularity of its derivatives, or else repeat all the arguments from ([7]), ([5]), or ([12]), carefully estimating all the constants that occur in the course of the proof.

Received
4 III 1964

REFERENCES

1. I. M. Gel' fand, G. E. Shilov, *Generalized functions and operations on them*, 2nd ed., Moscow, 1959.

2. V. V. Grushin, UMN, **16**, no. 4 (100), 147 (1961).

3. L. Hörmander, *On the theory of general partial differential operators*, IL, 1959.

4. L. Hörmander, Collection of translations, *Mathematics*, **7**, no. 1, 66 (1963).

5. L. Hörmander, Comm. Pure and Appl. Math., **11**, 2, 197 (1958).

6. T. Kasuga, Proc. Japan Acad., **37**, no. 2, 85 (1961).

7. B. Malgrange, Bull. Soc. Math. France, **85**, 283 (1957).

8. F. Trèves, Ann. Inst. Fourier, **9**, 1 (1959).

9. L. Schwartz, *Théorie des distributions*, vol. 1, 2, Paris, 1950.

10. G. E. Shilov, UMN, **14**, no. 5 (89), 3 (1959).

11. G. E. Shilov, Izv. vyssh. uchebn. zaved., matem., no. 2 (21), 160 (1961).

12. S. Mizohata, J. Math. Kyoto Univ., **1**, no. 3, 411 (1962).

Note: Figure translations are in progress. See original paper for figures.