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Abstract

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MATHEMATICS

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Investigation of a Class of Nonlinear Singular Integral Equations with Cauchy Kernel in the Class of Functions Vanishing at the Endpoints

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In the paper (^{1}) we established existence and uniqueness theorems for a bounded solution of the equation

$$u(x) = \lambda \int_a^b \frac{K[x, s, u(s)]}{s - x}\, ds \tag{1}$$

in the Hölder space $H_{K,\delta}$ (even if $K(x, s, u)$ has power growth). The elements of the space $H_{K,\delta}$ on the segment $[a, b]$ satisfy the conditions

$$|u(x)| \le K, \qquad |u(x + \Delta x) - u(x)| \le K|\Delta x|^\delta,$$

where $K = \text{const}$, $0 < \delta < 1$.

The purpose of the present note is to prove the theorem of existence and uniqueness of the solution of the equation

$$u(x) = \lambda q(x) \int_a^b \frac{f[u(s)]}{s - x}\, dS \tag{2}$$

in the class $H^0_{M,\delta}$, whose elements on the segment $[a, b]$ satisfy the conditions

$$|u(x)| \le Ml(x), \tag{3}$$

$$|u(x + \Delta x) - u(x)| \le M|\Delta x|^\delta, \tag{4}$$

where $M = \text{const}$, $l(x) = (x-a)^{\delta}(b-x)^{\delta}$, $0 < \delta < 1$, $q(x) = (x-a)^{\delta_1} \times (b-x)^{\delta_1}$, $0 < \delta < \delta_1 < 1$.

In $H^0_{M,\delta}$ the metric is introduced:

$$\rho_{C(l_1)}(u, v) = \max_{x\in[a,b]} l_1(x)|u(x) - v(x)|,$$

$$l_1(x) = [(x-a)(b-x)]^{-\delta'}, \qquad 0 < \delta' < \delta. \tag{5}$$

We note that $H^0_{M,\delta}$ is a closed, convex, and compact set in the sense of the metric $C(l_1)$. In addition, $H^0_{M,\delta} \subset \mathcal{L}_p(\rho)$, and $H_{M,\delta}$ is complete in the sense of convergence in $\mathcal{L}_p(\rho)$.

The space $\mathcal{L}_p(\rho)$ consists of functions $u(x)$ for which

$$\int_a^b \rho(x)|u(x)|^p\,dx < +\infty,$$

where $\rho(x) = [(x-a)(b-x)]^{-\delta' p}$, $1 < p < \delta a'^{-1}$.

Lemma 1. If on the interval $[-M(b-a)^{2\delta}, M(b-a)^{2\delta}]$ the function $f(u)$ satisfies the Lipschitz condition

$$|f(u_1) - f(u_2)| \le A|u_1 - u_2|, \tag{6}$$

then the function

$$w(x) = \int_a^b \frac{f[u(s)] - f(0)}{s - x}\,ds \tag{7}$$

for $x \in [a, b]$, $0 < \Delta x < \min\left(\frac{|x-a|}{4}, \frac{|x-b|}{4}\right)$, $u(x) \in H^0_{M,\delta}$, satisfies the conditions

$$|W(x)| \le MAL, \qquad |W(x+\Delta x) - W(x)| \le MAL|\Delta x|^{\delta}, \tag{8}$$

where $L = \text{const}$, independent of M and A.

Lemma 2. Under condition (6), the operator

$$Bu = q(x)\int_a^b \frac{f[u(s)] - f(0)}{s - x}\,ds \tag{9}$$

maps $H^0_{M,\delta}$ into $H^0_{M',\delta}$, where $M' = MC$,

$$C = \max\{AL(b-a)^{2(\delta_1-\delta)},\ [(b-a)^{2\delta_1-\delta} + (b-a)^{2\delta}]AL\}.$$

Lemma 3. Under condition (6), the operator B is continuous in the sense of the metric $C(l_1)$.

Consider the operator

$$Ku = \lambda q(x) \int_a^b \frac{f[u(s)]}{s-x}\, ds = \lambda Bu + \lambda f(0)\, q(x) \ln \frac{b-x}{x-a}. \tag{10}$$

Since

$$\left| [(x-a)(b-x)]^{-\delta} \ln \frac{b-x}{x-a} \right| < L',$$

then, by Lemma 2, we have

$$|v(x)| = |Ku| \le |\lambda|(MC + L'L'')\, l(x), \tag{11}$$

$$|v(x+\Delta x) - v(x)| \le |\lambda|(MC + L''L''')|\Delta x|^\delta, \tag{12}$$

where $L'' = |f(0)|$, and L''' is the Hölder constant for the function $q(x)\ln\frac{b-x}{x-a}$. From the continuity of the operator B in the sense of $C(l_1)$, it is easy to obtain the continuity of the operator K in the same sense.

Thus, applying Schauder' s principle, the following is established.

Theorem 1. If on $[-M(b–a)^{2\delta},\ M(b–a)^{2\delta}]$ the function $f(u)$ satisfies condition (6), then there exists a number

$$\lambda_0 = \min\left(\frac{M}{MC + L'L''}, \frac{M}{MC + L''L'''}\right),$$

such that for $|\lambda| < \lambda_0$ the nonlinear singular integral equation (2) has at least one solution

$$u(x) \in H^0_{M,\delta}.$$

Above we noted that the space $H^0_{M,\delta}$ is complete in the sense of the metric $\mathcal{L}_p(\rho)$. On the other hand, for the operator K it is easy to prove the validity of the inequality

$$\|Ku - Kv\| \le |\lambda|(b-a)^{2\delta_1} AF\|u-v\|_{L_p(\rho)}, \tag{13}$$

where F is the norm of a certain linear singular operator in the sense of $\mathcal{L}_p(\rho)$ (2). Consequently, the following is true.

Theorem 2. If $f(u)$ satisfies the conditions of Theorem 1 and condition (13), then for

$$|\lambda| < \min\left(\lambda_0,\ \frac{1}{(b-a)^{2\delta_1}AF}\right)$$

equation (2) has a unique solution $\varphi(x)$ in the space $H^0_{M,\delta}$. The successive approximations will converge in the metric of the space $\mathcal{L}_p(\rho)$.

We shall now establish the nature of the convergence of the successive approximations. Let

$$d_n = \left(\int_a^b \rho(x)\,|u_n(x) - \varphi(x)|^p\,dx\right)^{1/p},$$

where

$$u_n(x) = \lambda q(x)\int_a^b \frac{f[u_{n-1}(s)]}{s-x}\,ds,$$

$\varphi(x)$ is the solution of equation (2). We have $u_n(x) \in H^0_{M,\delta}$ and $d_n \to 0$ as $n \to 0$. Noting this, construct the set

$$E_n(x_0) = \left[x_0 - \frac{\sqrt{d_n}}{2},\ x_0 + \frac{\sqrt{d_n}}{2}\right],$$

where x_0 is an arbitrary point of the interval (a,b). Note that $E_n(x_0) \subset (a,b)$. The inequality

$$\left|\frac{u_n(\xi_n) - \varphi(\xi_n)}{(\xi_n - a)^{\delta'}(b-\xi_n)^{\delta'}}\right| \le d_n^{1-1/2p}, \tag{14}$$

is easily established, where ξ_n is some point of $E_n(x_0)$. Introduce the notation:

$$u_n^*(x) = [(x-a)(b-x)]^{-\delta'}u_n(x), \qquad \varphi^*(x) = [(x-a)(b-x)]^{-\delta'}\varphi(x).$$

Since $\varphi(x)$ and $u_n(x)$ belong to $H^0_{M,\delta}$, by virtue of (3) (see p. 25),

$$[(x-a)(b-x)]^{-\delta'}u_n(x) = u_n^*(x) \in H_{M'',\,\delta-\delta'}.$$

We may now estimate the differences:

$$\left|(x_0-a)^{-\delta'}(b-x_0)^{-\delta'}[u_n(x_0)-\varphi(x_0)]\right| \leq$$

$$\leq |u_n^*(x_0)-u_n^*(\xi_n)|+|u_n^*(\xi_n)-\varphi^*(\xi_n)|+|\varphi^*(x_0)-\varphi^*(\xi_n)|,$$

or

$$\left|[(x_0-a)(b-x_0)]^{-\delta'}[u_n(x_0)-\varphi(x_0)]\right| \leq 2M'' d_n^{(\delta-\delta')/2}+d_n^{1-1/2p}. \tag{15}$$

Since x_0 is an arbitrary point of $[a,b]$, from (15) we obtain that the sequence $\{u_n(x)\}$ converges to $\varphi(x)$ in the metric $C(l_1)$.

Thus the following has been proved.

Theorem 3. Convergence of a sequence of elements of $H^0_{M,\delta}$ in the metric $\mathcal{L}_p(\rho)$ entails convergence of the same sequence in the metric $C(l_1)$.

From Theorems 2 and 3 it follows:

Theorem 4. Under the conditions of Theorem 2, the unique solution in $H^0_{M,\delta}$ of equation (2) can be found by the method of successive approximations. The successive approximations converge in the metric $C(l_1)$.

Remark. These results are valid for more general equations

$$u(x)=\lambda q(x)\int_a^b \frac{f[x,s,u(s)]}{s-x}\,ds,$$

$$u(x)=\lambda F[x,w(x)],$$

where

$$w(x)=q(x)\int_a^b \frac{f[x,s,u(s)]}{s-x}\,ds.$$

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REFERENCES

[1] A. I. Guseinov, Kh. Sh. Mukhtarov, *DAN*, **146**, No. 2 (1962). [2] B. V. Khvedelidze, *Tr. Tbilissk. matem. inst.*, **23** (1956). [3] A. A. Babaev, *Uch. zap. Azerb. gos. univ. im. S. M. Kirova*, ser. phys.-math. and chem. sciences, No. 1 (1961).

Note: Figure translations are in progress. See original paper for figures.

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