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**MATHEMATICS**

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# FOLIATIONS OF CODIMENSION 1

*(Presented by Academician L. S. Pontryagin on 16 III 1964)*

Let us recall that in the author's preceding paper ([2]) a number of notions relating to the theory of smooth foliations (Pfaff systems) of codimension 1 on smooth compact manifolds with or without boundary were introduced and studied. For a sufficiently detailed bibliography of works on the theory of foliations, see ([1,2]). As in ([2]), here we shall study only smooth nonsingular oriented foliations of codimension 1 on orientable manifolds, and we shall not redefine the notions of orientation of a foliation, of the connected component of an orientable foliation, of the semigroups $t(A)$ of closed transversals passing through the leaf $A$, of the groups $P_j(A)$ and $N_j(A)$, $j = 1, 2$, of limit and non-limit cycles from the right (left) on the leaf $A$, and others. The main result of ([2]) was Theorem 2, stating that every nonsingular 2-foliation on a three-dimensional manifold $M^3$ with finite group $\pi_1(M^3)$ has a closed leaf–the torus $T^2 \subset M^3$, bounding a solid torus $D^2 \times S^1 \subset M^3$ with a Reeb foliation on $D^2 \times S^1$ (and even has a nested system of such solid tori). It goes without saying that the term "Reeb foliation" on $D^2 \times S^1$ is understood in a broad sense, referring only to the qualitative geometric character of the picture and not tying the definition of a Reeb foliation to any definite rate of convergence of noncompact leaves to the boundary compact leaf, as in ([1]).

Let us introduce some new notions. Observe that a mapping of a sphere with a marked point $f : S^i \to A$ into a leaf $A$ of a foliation on $M^n$ can be shifted to a nearby leaf to the right and to the left when $i > 1$, while for $i = 1$ one can shift the mappings $S^1 \to A \subset M^n$ representing elements of $N_1(A)$ (to the right) and of $N_2(A)$ (to the left). Moreover, under such a shift, mappings homotopic on $A$ will remain homotopic on a sufficiently nearby leaf, although nonhomotopic ones may become homotopic. We define the "limit epimorphisms"

$$\pi^{(j)}_{\lim}(A) : \pi_i(A) \to \pi^{(j)}_i(A), \qquad i > 1,$$

$$\pi^{(j)}_{\lim}(A) : N_j(A) \to \pi^{(j)}_1(A), \qquad i = 1,$$

whose kernel is generated by the homotopy classes of mappings $S^i \to A$ that are homotopic to zero under any shift to a nearby leaf on the right for $j = 1$, or on the left for $j = 2$.

**Theorem 1.** *Suppose that one of the following conditions is satisfied:*

a) *there is a closed transversal of the foliation that is homotopic to zero in the manifold $M^n$;*

b) *for some leaf $A \subset M^n$, the homomorphism of the embedding $\pi_1(A) \to \pi_1(M^n)$ is not a monomorphism;*

c) *the group $\pi_2(M^n)$ is nontrivial, and for all leaves $A \subset M^n$ the groups $\pi_2(A)$ are trivial.*

*Then there exists a leaf $A_0 \subset M^n$ such that the homomorphism*

$$\pi^{(j)}_{\lim}(A_0) : N_j(A_0) \to \pi^{(j)}_1(A_0)$$

*has a nontrivial kernel for $j = 1$ or 2.*

This theorem generalizes Lemma 2 of ([2]), which was connected only with item a). If $n = 3$ and, for some leaf $A \subset M^3$, the group $\pi_2(A^3)$ is nontrivial,

then $A = S^2$ and $M^3 = S^2 \times S^1$ or $S^2 \times I$, according to a theorem of Reeb and the orientability of the manifold and the foliation.

The following theorem is a development of Lemma 3 of paper 2.

**Theorem 2.** *If for a leaf $A_0 \subset M^3$ the group $\operatorname{Ker} \pi^{(j)}_{\lim}(A_0) \subset N_j(A_0)$ is nontrivial, then the leaf $A_0$ is compact, is a torus $T^2 \subset M^3$, homologous to zero in $M^3$, and one of the films bounded by it is homeomorphic to $D^2 \times S^1$, with the Reeb foliation on $D^2 \times S^1$.*

**Remark.** Theorem 2 is also true for $n > 3$, if $N_j(A_0)$ is replaced by $\pi_{n-2}(A_0)$ and one additionally requires that some element $\alpha \in \operatorname{Ker} \pi^{(j)}_{\lim}(A_0)$ be realized by an embedded sphere $S^{n-2} \subset A_0$.

The proof of Theorem 2 breaks up into a number of steps. First it is proved that one can find a nearby leaf $A'_0$ with a curve $\alpha_0 \in \operatorname{Ker} \pi^{(j)}_{\lim}(A'_0)$, onto whose images, after shifts to the right (or to the left), regular disks are spanned on their leaves. Then the family of these disks is analyzed. It turns out that this family determines a regular mapping $F : D^2 \times R^+ \to M^3$, where the maps $F/D^2 \times t$, $0 \le t < \infty$, coincide with the disks spanned on the images of the curve $\alpha_0$ after shifts to the right (or to the left), and the images $F/\partial D^2 \times t$ converge to the curve $\alpha_0$ as $t \to \infty$. It is proved that the curve $\alpha_0$ does not belong to the image $F(D^2 \times R^+)$. From all this one obtains the fact that through a point on the curve $\alpha_0$ of the leaf $A'_0$ there passes no closed transversal of the foliation, and therefore this leaf is compact. The remaining part of the proof is then carried out more or less simply.

From Theorems 1 and 2 it follows:

**Corollary 1.** *If an orientable foliation on an orientable manifold $M^3$ has no connected components homeomorphic to the Reeb foliation on $D^2 \times S^1$, then either $M^3 = S^2 \times S^1$ or $S^2 \times I$ and the foliation is trivial, or the universal cover $\widehat{M}^3$ of the manifold $M^3$ is contractible and the covering leaves on $\widehat{M}^3$ are planes.*

As is known, Haefliger proved that on manifolds with finite fundamental group there exist no nonexceptional analytic foliations of codimension 1 (see (1)). We apply the results of Theorems 1 and 2 to the question of the existence of analytic foliations on three-dimensional manifolds. There is the following essential

**Lemma.** *If on a three-dimensional manifold $M^3$ there exists an analytic 2-foliation, then on this manifold there exists a smooth 2-foliation without connected components homeomorphic to the Reeb foliation on $D^2 \times S^1$.*

The point is that if an analytic 2-foliation has a Reeb connected component, then this component, roughly speaking, is the result of "turbulization." Using this consideration, one can "deturbulize" the foliation, although in doing so we lose analyticity.

From the lemma and Corollary 1 it follows:

**Corollary 2.** *If on a manifold $M^3$ there exists an analytic foliation, then either $M^3 = S^2 \times S^1$ or $S^2 \times I$, or the universal cover $\widehat{M}^3$ of the manifold $M^3$ is contractible.*

In connection with the result of Corollary 2 it is interesting to note that smooth foliations exist on any orientable compact three-dimensional manifold without boundary, as Lickorish showed (unpublished). He also indicated a whole series of nontrivial foliations on $S^3$, having only one compact leaf—a knotted torus $T^2 \subset S^3$, where the axis of the torus is a nontrivial knot in $S^3$ with fundamental group having a finitely generated commutator subgroup.

We indicate one easily proved fact concerning the case of arbitrary $n \geq 3$.

**Theorem 3.** *Suppose that for a foliation on a closed manifold $M^n$ all the groups $P_j(A)$ are trivial for $j = 1$ and $j = 2$. Then the embedding $\pi_1(A) \to \pi_1(M^n)$ for all leaves $A$ is a monomorphism onto a normal*

divisor, the quotient group $\pi_1(M^n)/\pi_1(A)$ is free abelian and $\hat{M}^n = \hat{A} \times R$, where $\hat{M}^n$ and $\hat{A}$ denote, respectively, the universal covering manifolds of $M^n$ and of the leaf $A$.

The proof of the theorem is not very difficult and splits into two steps. First, using the absence of limit cycles on the leaves, one proves that the normal to the foliation on $\hat{M}^n$ intersects all leaves on $\hat{M}^n$, and each leaf exactly once. Using this result, one constructs a representation of the group $\pi_1(M^n)$ in the group $\operatorname{diff} R$ of diffeomorphisms of the line, whose kernel turns out to be precisely $\pi_1(A)$. The image of this representation consists of transformations $R \to R$

having no fixed points, whence it follows that the group $\pi_1(M^n)/\pi_1(A)$ is free abelian by a theorem of Hölder.

From Theorem 3 it follows, for example, that if all leaves are Euclidean spaces $R^{n-1}$, then the manifold $M^n$ has the homotopy type of the torus $T^n$.

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*Note added in proof.* As V. I. Arnold pointed out to me, from Theorem 1 one can derive that rank $M^n \leq n-2$, $n \geq 3$, if $\pi_2(M^n)$ is finite or $\pi_2(M^n) \neq 0$, where the manifold $M^n$ is closed (the rank of $M^n$ is the maximal number of linearly independent pairwise commuting vector fields). For example, rank $S^3 = 1$, rank $S^2 \times T^{n-2} = n-2$.

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## CITED LITERATURE

1. A. Haefliger, Ann. Scuola Norm. Sup. di Pisa, ser. III, **16**, F. IV, 367 (1962).
2. S. P. Novikov, DAN, **155**, No. 5 (1964).

*Note: Figure translations are in progress. See original paper for figures.*