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Abstract

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PHYSICS

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THE LAGRANGIAN OF A CONTINUOUS MEDIUM IN RIEMANNIAN SPACE

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The Hamilton–Jacobi equation for a material point in Riemannian space (in the general theory of relativity), as is well known, has the form (^{1a}, §85)

$$g^{ik}\frac{\partial s^*}{\partial x^i}\frac{\partial s^*}{\partial x^k} = -m^2c^2 = -\frac{\bar{E}^2}{c^2}, \tag{1}$$

where $\bar{E}$ is the energy of the particle; s^* is the action;

$$\frac{\partial s^*}{\partial x^i} = p_i = mcu_i \tag{2}$$

is the 4-momentum of the particle;

$$\frac{\partial s^*}{\partial x^\alpha} = p_\alpha = \frac{ma_\alpha}{\sqrt{1-a^2/c^2}} = \frac{ma_\alpha}{\theta} \tag{3}$$

is the 3-momentum;

$$c\partial s^*/\partial x^0 = E^* \tag{4}$$

is the total energy of the particle.

In the case of a continuous medium, the energy attributable to one particle (^{2}),

$$\bar{E} = (\partial e/\partial n)_\sigma = mw, \tag{5}$$

where w is the heat content of a unit of rest mass (including internal energy), σ is entropy, and n is the density of the number of particles.

In this case equation (1) can be written in the form

$$g^{ik}\frac{\partial s^*}{\partial x^i}\frac{\partial s^*}{\partial x^k} = -m^2\frac{w^2}{c^2}. \tag{6}$$

Introducing actions calculated per unit mass, $s = s^*/m$, we finally write equation (6), which is the Hamilton–Jacobi equation for a continuous medium, in the form

$$g^{ik}\frac{\partial s}{\partial x^i}\frac{\partial s}{\partial x^k} + \frac{w^2}{c^2} = 0, \tag{7}$$

with

$$c\,\partial s/\partial x^i = wu_i. \tag{8}$$

The Lagrangian in the mechanics of continuous media is the pressure ([3,4])

$$p = L = (w - E)/v, \tag{9}$$

where $v = 1/mn$ is the specific volume.

From (7) and (9) we find that

$$L = p = \frac{1}{v}\left[-E + ic\sqrt{g^{ik}s_i s_k}\right], \tag{10}$$

where, to shorten the calculations, $s_i = \partial s/\partial x^i$ has been introduced.

First of all, let us find the energy-momentum tensor for the Lagrangian (10). As is known (([16]), § 94),

$$\frac{\sqrt{-g}}{2}T_{ik} = -\frac{\partial(\sqrt{-g}L)}{\partial g^{ik}} + \frac{\partial}{\partial x^l}\frac{\partial(\sqrt{-g}L)}{\partial g^{ik}_{/\partial x^l}} + \cdots \tag{11}$$

Here the quantities s_i are regarded as constant.

If one takes into account that

$$dE = T\,d\sigma - p\,dv \tag{12}$$

and that for adiabatic processes $d\sigma = 0$, then the calculations give

$$T_{ik} = (p + \varepsilon)u_i u_k + g_{ik}p. \tag{13}$$

The same expression can be obtained if one uses another variational equation of Lagrange:

$$\sqrt{-g}\,T_i^k = \sqrt{-g}\,L\delta_i^k - \frac{\partial s}{\partial x^i}\frac{\partial(\sqrt{-g}L)}{\partial\,\partial s/\partial x^k}. \tag{14}$$

Here the quantities g^{ik} are regarded as constant.

The field equations have the form

$$\frac{\partial(\sqrt{-g}L)}{\partial s} - \frac{\partial}{\partial x^k}\frac{\partial(\sqrt{-g}L)}{\partial s_k} = 0. \tag{15}$$

The calculations give

$$\frac{\partial}{\partial x^k}\left[\sqrt{-g}\,\frac{cg^{ik}s_i}{vw}\right] = \frac{\partial}{\partial x^k}\left[\sqrt{-g}\,\frac{u_k}{v}\right] = 0, \tag{16}$$

which is the equation of continuity. The quantities $\dot{s}$ and $1/vw = \delta$ are canonically conjugate.

For a medium obeying the equation of state

$$pv^k = A = \text{const}, \tag{17}$$

$$w = \alpha c^2 + \frac{k}{k-1}\,A^{1/k}p^{(k-1)/k}, \tag{18}$$

where $\alpha = 1, 0$, respectively, for the relativistic and ultrarelativistic cases. In this case

$$L = \text{const}\,\left[i\sqrt{g^{ik}s_i s_k} - \alpha c\right]^{k/(k-1)}. \tag{19}$$

(Equation (11) again gives the correct field tensor, which is quite natural.)

The field equations lead to an important equation, convenient for solution:

$$g^{kr}s_r s_l\left[g^{il}s_{ik} + \frac{1}{2}s_i\frac{\partial g^{il}}{\partial x^k}\right]\left[\frac{2-k}{k-1}\sqrt{-g^{mn}s_m s_n} + \alpha c\right] +$$

$$+\left[g^{kl}s_{kl} + s_l\frac{\partial g^{kl}}{\partial x^k}\right]g^{ir}s_i s_r\left[\sqrt{-g^{mn}s_m s_n} - \alpha c\right] +$$

$$+\frac{\partial \ln\sqrt{-g}}{\partial x^k} s_r s_i s_m g^{kr} g^{im} \left[\sqrt{-g^{mn} s_m s_n} - \alpha c\right] = 0. \tag{20}$$

The particular case of motion of an ultrarelativistic gas ($\alpha > 0$) is easily investigated:

$$(s_i s^i)^{\frac{2-k}{2(k-1)}}\, s^k \frac{\partial \ln\sqrt{-g}}{\partial x^k} + \frac{\partial}{\partial x^k}\left[s^k (s_i s^i)^{\frac{2-k}{2(k-1)}}\right] = 0. \tag{21}$$

In the particular, but most important and natural case $k = 4/3$, then

$$s_j s^j s^k \frac{\partial \ln\sqrt{-g}}{\partial x^k} + \frac{\partial}{\partial x^k}(s_i s^i s^k) = 0. \tag{22}$$

For $k = 2$ we have $s^k \frac{\partial \ln\sqrt{-g}}{\partial x^k} + \frac{\partial s^k}{\partial x^k} = 0$, or

$$\frac{\partial\left(\sqrt{-g}\, s^k\right)}{\partial x^k} = 0. \tag{23}$$

The results obtained are easily generalized to the case of motion of charged particles of a continuous medium in an electromagnetic field. In this case, as is well known, for a particle ((1^a), §87)

$$\frac{\partial s^*}{\partial x^i} = mcu_i + \frac{e}{c}A_i, \quad \left(\frac{\partial s^*}{\partial x^i} - \frac{e}{c}A_i\right)\left(\frac{\partial s^*}{\partial x^k} - \frac{e}{c}A_k\right) g^{ik} + m^2c^2 = 0, \tag{24}$$

where A_i are the components of the electromagnetic-field vector.

For a continuous medium, analogously, we shall have

$$cs_i = c\frac{\partial s}{\partial x^i} = \omega u_i + \frac{e}{m}A_i, \tag{25}$$

where m is the total mass of the particle. Further we have

$$\frac{\omega^2}{c^2} = -\left(s_i - \frac{e}{mc}A_i\right)\left(s^i - \frac{e}{mc}A^i\right). \tag{26}$$

It is now easy to verify that all the equations derived above are generalized to the case of an electromagnetic field by the simple replacement in them

$$s_i \to \hat{s}_i = s_i - \frac{e}{mc}A_i. \tag{27}$$

Let us verify the relations obtained.

The conservation laws for matter together with the electromagnetic field are given by the equations

$$\hat{T}^k_{i;k} = \left(T^k_i + \overline{T}^k_i\right)_{;k} = 0, \tag{28}$$

where

$$\overline{T}^k_i = \frac{1}{4\pi}\left[F_{il}F^{kl} - \frac{1}{4}\delta^k_i F_{lm}F^{lm}\right] \tag{29}$$

is the energy-momentum tensor of the electromagnetic field.

Write (28) in the form:

$$T^k_{i;k} = -\overline{T}^k_{i;k} = f_i, \tag{30}$$

where f_i is the 4-force of interaction of matter with the electromagnetic field. We compute

$$T^k_{i;k} = \frac{1}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}\,T^k_i\right)}{\partial x^k} - \frac{T^{kl}}{2}\frac{\partial g_{kl}}{\partial x^i} = f_i. \tag{31}$$

Multiplying scalarly by u^i and using the thermodynamic condition

$$d\omega = v\,dp + T\,d\sigma, \tag{32}$$

we find

$$u^k\left[\frac{\partial(\omega u_i)}{\partial x^k} - \frac{\partial(\omega u_k)}{\partial x^i}\right] = vf_i + T\,\frac{\partial\sigma}{\partial x^i}. \tag{33}$$

Let us now compute $\overline{T}^k_{i;k} = -f_i$. Simple calculations show that

$$\overline{T}^k_{i;k} = -f_i = -\frac{1}{c}F_{ik}j^k,$$

where

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}.$$

The components of the electromagnetic field and of the current vector are related by Maxwell’ s equations

$$\frac{\partial F_{ik}}{\partial x^l}+\frac{\partial F_{li}}{\partial x^k}+\frac{\partial F_{kl}}{\partial x^i}=0,\qquad \frac{4\pi}{c}j^i=\frac{1}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}F^{ik}\right)}{\partial x^k},$$

where the obvious continuity equation for the current holds, $\partial(\sqrt{-g}j^i)/\partial x^i=0$. In this case (33) takes the form

$$\frac{d(wu_i)}{ds}+\frac{\partial w}{\partial x^i}-\frac{w}{2}u^k u^l\frac{\partial g_{kl}}{\partial x^i}=-\frac{v}{c}F_{ik}j^k+T\frac{\partial\sigma}{\partial x^i}. \tag{34}$$

If we consider processes without energy dissipation, then

$$f_i u^i=0;\qquad d\sigma/ds=0;\qquad \sigma=\text{const}.$$

Then

$$j^k=\frac{\delta c}{\sqrt{-g}}\frac{dx^k}{dx^0}=\delta c u^k\frac{ds}{\sqrt{-g}\,dx^0}=\delta c\frac{u^k}{\sqrt{-g}\,u^0},$$

where $u^k=dx^k/ds$;

$$-\frac{v}{c}F_{ik}j^k=-\frac{e}{m}F_{ik}u^k,$$

with $\delta v/\sqrt{-g}\,u^0=e/m$,

$$vf_i=-v\overline{T}^k_{i;k}=-\frac{v}{c}F_{ik}j^k=-\frac{e}{m}F_{ik}u^k=\frac{e}{m}u^k\left[\frac{\partial A_k}{\partial x^i}-\frac{\partial A_i}{\partial x^k}\right].$$

In this case (33) takes the form:

$$u^k\left[\frac{\partial\left[wu_i+\frac{e}{m}A_i\right]}{\partial x^k}-\frac{\partial\left[wu_k+\frac{e}{m}A_k\right]}{\partial x^i}\right]=0, \tag{35}$$

whence

$$c\frac{\partial s}{\partial x^i}=wu_i+\frac{e}{m}A_i,$$

and we arrive at the equation (25) found above. Let us now calculate the quantities

$$\frac{\partial(\sqrt{-g}L)}{\partial x^0} = \frac{\partial(\sqrt{-g}L)}{\partial g^{lm}}\frac{\partial g^{lm}}{\partial x^i} + \sqrt{-g}\,\frac{\partial L}{\partial s_k}\frac{\partial s_k}{\partial x^i},$$

transforming and using the field equations (15), we find that

$$\frac{\partial(\sqrt{-g}T_i^k)}{\partial x^k} - \frac{T^{lm}}{2}\frac{\partial g_{lm}}{\partial x^i} = T^k_{i;k} = 0,$$

i.e. we find in a natural way the equation of conservation of energy-momentum.

In the case of a medium obeying the equation of state $p = (k-1)\varepsilon$ (an ultrarelativistic gas), equations (35) and (25), with the replacement $w \to c^2(p/p_0)^{\frac{k-1}{k}}$, where p_0 is some initial pressure, will hold also for $\sigma \neq \text{const}$, but under the condition $d\sigma/ds = 0$, i.e. not only for isentropic, but also for adiabatic motions (^{5}).

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Note: Figure translations are in progress. See original paper for figures.