



Soviet-era science, translated into English

MATHEMATICS

B. EFIMOV

1964

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196401.07179>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

B. EFIMOV

UNIMODULAR WEIGHT FUNCTIONS AND THE PROBLEM OF P. S. ALEXANDROV AND P. S. URYSOHN IN THE THEORY OF BICOMPACTS

(Presented by Academician P. S. Alexandrov, 9 V 1964)

§ 1. **Weight characteristics and weight functions.** As early as in the memoir ⁽¹⁾ P. S. Alexandrov and P. S. Urysohn showed that in the theory of topological spaces an important role is played by the so-called weight characteristics. Such are, for example, the weight (wX) of a space X or its density (sX)—the least cardinality of an everywhere dense subset of X . In the present paper we study weight characteristics that assign to each topological space X a cardinal number $\mathfrak{n}X$ satisfying the following five conditions:

Q_1 . If X is homeomorphic to Y , then $\mathfrak{n}X = \mathfrak{n}Y$.

Q_2 . If U is an open subset of X , then $\mathfrak{n}U \leq \mathfrak{n}X$.

Q_3 . If $X = U_1 \cup \dots \cup U_k$ and each U_i , $i = 1, \dots, k$, is open in X , then

$$\mathfrak{n}X \leq \mathfrak{n}U_1 + \dots + \mathfrak{n}U_k.$$

Q_4 . If

$$X = \bigcup_{\alpha \in A} U_\alpha,$$

where each U_α is open in X and all the U_α are disjoint, then

$$\mathfrak{n}X \geq \sum_{\alpha \in A} \mathfrak{n}U_\alpha.$$

Q_5 . $\mathfrak{n}X = 0$ if and only if $X = \emptyset$.

Besides the aforementioned wX and sX , these conditions are satisfied by such weight characteristics as the Kuratowski number cX ⁽²⁾, the proper weight and the calibers of N. A. Shanin ^(3, 4), the combinatorial weight of Yu. M. Smirnov ⁽⁵⁾, and many others. We note that among all weight characteristics defined

on a topological space X there exists a least one. Such a characteristic is the Kuratowski number cX . Indeed,

$$cX = \sup_{\alpha} |\mathfrak{A}_{\alpha}|^*,$$

if \mathfrak{A}_{α} is a certain system of disjoint nonempty open subsets of X , the supremum being taken over all such systems. Put

$$V_{\alpha} = \bigcup_{\beta \in B} U_{\beta}^{\alpha},$$

if $\mathfrak{A}_{\alpha} = \{U_{\beta}^{\alpha}\}$, $\beta \in B$. Then, by conditions Q_4 and Q_5 , we obtain

$$\mathfrak{n}V_{\alpha} \geq \sum_{\beta \in B} \mathfrak{n}U_{\beta}^{\alpha} \geq |B| = |\mathfrak{A}_{\alpha}|.$$

Since $V_{\alpha} \subset X$, by Q_2 we have $\mathfrak{n}X \geq \mathfrak{n}V_{\alpha} \geq |\mathfrak{A}_{\alpha}|$, and this inequality is valid for all α . Hence it follows that

$$\mathfrak{n}X \geq \sup_{\alpha} |\mathfrak{A}_{\alpha}| = cX.$$

Let $\mathfrak{n}X$ be some weight characteristic of the space X satisfying conditions Q_1 - Q_5 . Let $\mathfrak{B} = \{O_{\alpha}x\}$, $\alpha \in A$, be some fundamental system of neighborhoods of a point $x \in X$. The cardinal number

$$\mathfrak{n}(x, X) = \inf_{\alpha \in A} (\mathfrak{n}O_{\alpha}x)$$

will be called the **local characteristic** of the point x in the space X . For example, if $\mathfrak{n}X = wX$, then $\mathfrak{n}(x, X) = w(x, X)$ is called the local weight ⁽⁶⁾ of the point x in X ; if $\mathfrak{n}X = sX$, then $\mathfrak{n}(x, X) = s(x, X)$ is the local density, etc. It is easy to see that $\mathfrak{n}(x, X)$ does not depend on the choice of the fundamental system of the point x .

* The cardinality of a set X is denoted by $|X|$.

By a **weight function** we shall mean, first, a function defined on the set of points of a topological space X and assigning to each point $x \in X$ either the local characteristic $\mathfrak{n}(x, X)$ or the neighborhood character $\chi(x, X)$; second, a function defined on the set of closed subsets of the space X and assigning to each closed subset $F \subset X$ either the weight characteristic $\mathfrak{n}F$, or the neighborhood character $\chi(F, X)$. From the whole class of weight functions defined on the set of points of a space X , we single out the step weight functions. Namely, we shall call a function $\mathfrak{n}(x)$, $x \in X$, **stepwise** if in X there exists a disjoint system of open sets $\{U_{\alpha}\}$, $\alpha \in A$, such that $\left[\bigcup_{\alpha \in A} U_{\alpha}\right] = X$ and on each U_{α} the weight function is constant, i.e. $\mathfrak{n}(x, X) = \mathfrak{n}_{\alpha}$ for all $x \in U_{\alpha}$.

Theorem 1. The weight function $\mathfrak{n}(x)$ of any local characteristic $\mathfrak{n}(x, X)$ of a regular space X is stepwise.

Let us note that the weight function $\chi(x)$ is not stepwise even for bicompecta. For example, the Stone-Ćech compactification βQ of the rational points of the real line has everywhere dense sets $M_1 = Q$ and $M_2 = \beta Q \setminus Q$ such that $\chi(x, \beta Q) \leq \aleph_0$ for $x \in M_1$ and $\chi(x, \beta Q) \geq \aleph_1$ for $x \in M_2$.

A weight function $\mathfrak{n}(x)$ defined on the set of points of a space X , or on the set of its closed subsets, will be called **modular** if for every open $U \subset X$ we have $\mathfrak{n}[U] \leq \sup_x \mathfrak{n}(x, [U])$, $x \in [U]$.

Theorem 2. The weight function $\mathfrak{n}(x)$ of any local characteristic $\mathfrak{n}(x, X)$, defined on a bicompectum X , is modular.

A weight function $\mathfrak{n}(x)$ defined on the set of points of a space X will be called **unimodular** if for every open $U \subset X$ and every dense set $M \subset U$ we have $\mathfrak{w}[U] \leq \sup_{x \in M} \mathfrak{n}(x, [U])$.

Correspondingly, a weight function $\mathfrak{n}F$, defined on the set of closed subsets of X , will be called **unimodular** if for every open $U \subset X$ we have $\mathfrak{w}[U] \leq \sup_F \mathfrak{n}F$ for all $F \subset [U]$.

Theorem 3. The weight function $\chi(x)$, defined on a nondiscrete dyadic space, is unimodular.

Theorem 4. Every function of a weight characteristic, defined on the set of all closed subsets of a dyadic bicompectum R , is unimodular.

Theorem 5. If the function $\chi(x)$ on a regular topological space is unimodular and its range of values is discrete, then $\chi(x)$ is a stepwise function.

The proof of Theorem 3 is easy to obtain by using two theorems of the author (Theorem 3 from (7) and Theorem 3 from (8)). The proof of Theorem 4 is not simple. First of all it is proved that if $\mathfrak{w}R = \sup \chi(x, R)$, $x \in R$, then for every infinite cardinal number $\aleph_{p+1} \leq \chi(x_p, R)$ the space R topologically contains a discrete space T_p of cardinality \aleph_{p+1} . Next, the unimodularity of the function $c(F)$, $F \subset R$, is established. Hence, by the minimality of the weight function $c(F)$ among all weight functions $\mathfrak{n}(F)$, the unimodularity of any weight function follows. In the proof of Theorem 5, Theorem 1 plays an essential role. Theorem 5 is a far-reaching generalization of Theorem 1 from (6).

§ 2. A-classes. In the memoir (1), P. S. Alexandroff and P. S. Urysohn posed a problem that has remained unsolved up to now: does there exist a bicompectum with the first axiom of countability and of cardinality greater than the continuum? We shall now construct two classes of bicompecta for which this problem has a positive solution. More precisely, we shall call a class of bicompecta \mathfrak{B} an **A-class** if, for every $X \in \mathfrak{B}$, the assertion “ X satisfies

“the first axiom of countability” entails that “the cardinality of X does not exceed the cardinality of the continuum c .” We shall call an **A-class complete** if it contains all bicompecta of cardinality $\leq c$.

First class. Let $\mathfrak{G} = \{Y_\alpha\}$, $\alpha \in B$, be the set of all bicompacta of weight $\leq c$. Put

$$Z = \prod_{\alpha \in B} Y_\alpha.$$

We shall say that a bicompactum R belongs to the first class if there exists a cardinal number τ such that R is a continuous image of Z^τ .

We show that this class is an A -class. Let $R = f(Z^\tau)$ and $\sup \chi(x, R) \leq \aleph_0$, $x \in R$. Since

$$Z^\tau = \left(\prod_{\alpha \in B} Y_\alpha \right)^\tau = \prod_{\beta} \prod_{\alpha} Y_\alpha = \prod_{\alpha, \beta} Y_{\alpha, \beta},$$

then, applying Theorem 2 from the author's paper ⁽⁹⁾, we obtain

$$wR \leq \max \left[\sup_{\alpha, \beta} wY_{\alpha, \beta}, \sup_{x \in R} \chi(x, R) \right] = \max(c, \aleph_0) = c.$$

Let M be a dense subset of R and $|M| \leq c$. Since every point $x \in R$ is completely determined by a sequence $\{x_n\} \subset M$ converging to it, we have $|R| \leq |M|^{\aleph_0} \leq c^{\aleph_0} = c$, as required. The completeness of this class follows from its definition. It is just as easy to show that it is closed with respect to the direct product, taken in any number, and to continuous mappings.

Second class. Let

$$I^\tau = \prod_{\alpha \in B} I_\alpha$$

be the product of the intervals $I_\alpha = \mathfrak{E}\{0 \leq x \leq 1\}$. Denote by Σ the subset of I^τ consisting of those points $x = (x_\alpha) \in I^\tau$ for which $x_\alpha \neq 0$ for at most a countable set of coordinates. We shall say that a bicompactum R belongs to the second class if there exist a cardinal number τ and a closed set $F \subset \Sigma \subset I^\tau$ such that R is a continuous image of F .

We prove that this class is an A -class. Let $R = f(F)$, $F \subset \Sigma \subset I^\tau$, and $\sup_{x \in R} \chi(x, R) \leq \aleph_0$. Suppose that $|R| \geq \mathfrak{m} > c$, where \mathfrak{m} is the least cardinal number greater than c . We shall then show that R contains b_0T , the minimal compactification of a discrete space T of cardinality \mathfrak{m} . This will mean that $\chi(Z, R) \geq \mathfrak{m}$, if Z is the vertex of the bicompactum b_0T . We proceed to the construction of b_0T . Put $F_x = f^{-1}x$. From each F_x choose a point q_x . Denote the chosen set by M . Since $M \subset \Sigma$, for each point $q_x \in M$ the set $A_x \subset B$ of coordinates different from zero is at most countable. We shall call the set A_x

the base of the point q_x . Denote by S the system of all sets $\{A_x\}$. Observe that there exists a quasi-disjoint subsystem* $\sigma \subset S$ of cardinality $\geq \mathfrak{m}$. Indeed, otherwise the cardinality of every quasi-disjoint subsystem would be $\leq c$; then, by Michael's theorem⁽¹⁰⁾, we would have $|S| \leq c^{\aleph_0} = c$, since $|A_x| \leq \aleph_0$. Further, for each A_x there exist only $c^{\aleph_0} = c$ different points having A_x as their base. Hence finally $|M| \leq |S| \cdot c = c$, contrary to the supposition. Let σ be a quasi-disjoint system of cardinality $\geq \mathfrak{m}$. Two cases are possible: either $\bigcap A_x = \mathfrak{E} \neq \emptyset$, or $\bigcap A_x = \emptyset$. Since both are treated analogously, consider only the first case. Note that $|\mathfrak{E}| \leq \aleph_0$, and therefore the number of all possible real-valued functions defined on \mathfrak{E} does not exceed $c^{\aleph_0} = c$. Since $|\sigma| \geq \mathfrak{m}$, there exists a subsystem $\sigma_1 \subset \sigma$ of cardinality $\geq \mathfrak{m}$ such that, for all $A_x \in \sigma_1$, the values of the points q_x on the set \mathfrak{E} coincide. Consider the set of points $M' = \{q_x\} \subset M$, for which $A_x \in \sigma_1$. Since any index $\alpha \in B$ belongs either to one

* A system S of sets is called quasi-disjoint if every point $x \in \bigcap S$ belongs to at most one $s \in S$ ⁽¹⁰⁾.

A_x , or all together, then the set M' is discrete in the topology induced from I^τ . On the other hand, the point $t = \{t_\alpha\}$, defined as follows: $t_\alpha = 0$ if $\alpha \notin \mathfrak{E}$, and $t_\alpha = q_{x\alpha}$ if q_x is any point of M' , is the unique limit point of M' in I^τ . It is easy to see that $t \in \Sigma$; therefore, by the closedness of F , $t \in F$. Let $t \in F_x$. Remove from M' the point q_x , if it is contained in $F_x \cap M'$. Thus a bicomcompactum b_0T , $|T| \geq \mathfrak{m}$, $b_0T \subset F$, has been constructed, which is mapped one-to-one and continuously onto R . Consequently, on b_0T the mapping f is a homeomorphism and $f(b_0T)$ is the desired bicomcompactum. This means, as was noted above, that $|R| \leq c$. Thus the assertion is completely proved.

Theorem 6. *Every separable bicomcompactum (i.e. $sR \leq \aleph_0$) of the second class is metrizable.*

A system of real-valued functions $L = \{f_\alpha\}$, defined on a completely regular space X , will be called **pointwise countable** if, for each point $x \in X$, the set of all functions $f_\alpha(x) \neq 0$ is at most countable.

Theorem 7. *A bicomcompactum R is topologically contained in Σ if and only if there exists on R a pointwise countable system of functions separating all points of R .*

Consider the bicompacta A_1 and A_2 constructed by P. S. Aleksandrov and P. S. Uryson in (1). Since $sA_1 = \aleph_0$, while $wA_1 = c$, A_1 does not belong to the second class. On the other hand, the bicomcompactum A_2 belongs to this class, since there exists on it a pointwise countable system of functions separating all points. Hence it follows, first, that this class is not complete, and, second, that the first axiom of countability still does not imply metrizability for bicompacta of this class.

In conclusion, the author expresses gratitude to P. S. Aleksandrov for his attention to this work.

Moscow State University
named after M. V. Lomonosov

Received
29 IV 1964

REFERENCES

1. P. S. **Aleksandrov**, P. S. **Uryson**, Tr. Mat. Inst. im. V. A. Steklova, Acad. Sci. USSR, **31** (1950).
2. D. **Kurepa**, Publ. Math. Univ. Belgrad, **4** (1935).
3. N. A. **Shanin**, DAN, **38**, No. 5-6, 166 (1943).
4. N. A. **Shanin**, Tr. Mat. Inst. im. V. A. Steklova, Acad. Sci. USSR, **24**, 3 (1948).
5. Yu. M. **Smirnov**, UMN, **6**, issue 4, 204 (1951).
6. B. **Efimov**, Vestn. Mosk. Univ., Ser. Math., **2**, 3 (1964).
7. B. **Efimov**, DAN, **149**, No. 5, 1011 (1963).
8. B. **Efimov**, DAN, **151**, No. 5, 1021 (1963).
9. B. **Efimov**, DAN, **152**, No. 4, 794 (1963).
10. E. **Michael**, Proc. Am. Math. Soc., **13**, No. 2, 281 (1962).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.