

ON TOPOLOGICALLY REFLEXIVE $\backslash(KB\backslash)$ -SPACES

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Abstract

Full Text

MATHEMATICS

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ON TOPOLOGICALLY REFLEXIVE KB -SPACES

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It is known that many other Banach spaces which turn out to be reflexive* are closely connected with the space L of summable functions; for example, L_p for $p > 1$ and many Orlicz spaces. Recall that an Orlicz space, in the case when the functions are given on a bounded closed subset of Euclidean space, is reflexive if and only if the defining N -function $M(u)$ and the complementary N -function satisfy the Δ_2 -condition ((¹), p. 152).

In this note it is shown that the theory of semi-ordered spaces makes it possible to construct analogous, and even more varied, reflexive and at the same time KB -spaces, starting from an arbitrary KB -space. The notion of a KB -space, as is known, was introduced by L. V. Kantorovich. It is a K -space in which a norm is defined that turns it into a Banach space, and the following conditions are satisfied: 1) from $|x| \leq |y|$ it follows that $\|x\| \leq \|y\|$ (monotonicity of the norm); 2) if $x_n \downarrow 0$, then $\|x_n\| \rightarrow 0$; 3) if $x_n \uparrow +\infty$, then $\|x_n\| \rightarrow +\infty$ (see also (²)).

Let X be an arbitrary KB -space in which a complete set of pairwise disjoint positive elements has been distinguished; \hat{X} is the maximal extension of the space X (²). It makes sense (³) to speak of continuous functions defined on \hat{X} . Put, for arbitrary $p \geq 1$,

$$X_p = \{x : x \in \hat{X}, |x|^p \in X\},$$

i.e. X_p consists of all elements of \hat{X} whose p -th power of the modulus is contained in X . We introduce on X_p a norm by setting

$$\|x\|_p = \| |x|^p \|^{1/p},$$

where $\| \cdot \|$ is the norm in the original KB -space X .

Theorem 1. X_p for $p > 1$ is a reflexive KB -space.

We note that if $X = L$, then X_p coincides with the usual space L_p . Although Theorem 1 is a direct generalization of the known theorem on the reflexivity of the spaces L_p , nevertheless the proof of Theorem 1 is carried out by an entirely

different method and relies on certain considerations from the theory of semi-ordered spaces and on D. P. Milman's theorem on the reflexivity of uniformly convex spaces (it is not the spaces X_p themselves that turn out to be uniformly convex, but certain auxiliary spaces).

Let us also dwell on the spaces $L_{(\omega)}$, considered by Halperin, for example, in ⁽⁴⁾. If we put $X = L_{(\omega)}^1$ and if this X is a KB -space, then $X_p = L_{(\omega)}^p$ for $p > 1$ is a reflexive KB -space.

The spaces L^P , where $P = (p_1, p_2, \dots, p_n)$, considered in detail by A. Benedek and Panzone ⁽⁵⁾, are also closely connected with the spaces X_p introduced by us. If $1 < p_i < \infty$, where $i = 1, 2, \dots, n$, then L^P coincides with our space X_p , constructed from $X = L^Q$, where $p = \min p_i$ and $Q = (p_1/p, p_2/p, \dots, p_n/p)$. Since L^Q is a KB -space, it follows from this

* Throughout the note, the term reflexivity is understood in the sense of the theory of normed spaces.

(under the indicated conditions $1 < p_i < \infty$) the reflexivity of the spaces L^P follows.

Theorem 2. Let X be a separable continuous KB -space with unit, and let $M(u)$ be an N -function which, together with the complementary N -function, satisfies the Δ_2 -condition. Put

$$X_M = \{x : x \in X, M(x) \in X\}$$

and introduce on X_M a new norm by setting

$$\|x\|_{(M)} = \inf \left\{ K : K > 0, \left\| M\left(\frac{x}{K}\right) \right\| \leq 1 \right\}$$

(here $\| \cdot \|$ is the norm in X). Then X_M , with the indicated norm, is a reflexive separable KB -space.

Let us note that if, for example, $X = L[0, 1]$, then X_M is the Orlicz space L_M^* , equipped with the Luxemburg norm, equivalent to the Orlicz norm ⁽¹⁾.

The **proof** of Theorem 2 is based on known theorems on the concrete representation of separable KB -spaces ⁽³⁾.

Theorem 3. Let X be a KB -space with unit. Then for every $x \in X$ there exists a fundament* Y in X such that $x \in Y$ and such that on Y one can introduce a new norm under which Y will be a reflexive KB -space.

This theorem follows from Theorem 1. Namely, without loss of generality one may assume that x is a bounded element, since otherwise this could be achieved

by choosing a new unit. Then for Y one may take any of the spaces X_p with $p > 1$. $X_p \subset X$, since X is a space with unit.

The following is somewhat more difficult to prove.

Theorem 4. Let X be an infinite-dimensional KB -space with unit. Then for every $x \in X$ there exists a fundament Z in X such that $x \in Z$ and such that Z , with some new norm, is a non-reflexive KB -space.

From some results of ⁽³⁾ it is not hard to derive the following.

Let P be a KB -space with unit, on which an essentially positive linear functional f_0 is given. Put

$$Q = \left\{ x : x \in \hat{P}, \sup_{\substack{0 \leq x' \leq |x| \\ x' \in P}} f_0(x') < +\infty \right\},$$

where \hat{P} is the maximal extension of the space P . Denote by \bar{f}_0 the completely linear extension of the functional f_0 to all of Q . It exists and is unique ⁽³⁾. Put also

$$R = \{y : y \in Q, xy \in Q \text{ for every } x \in P\}.$$

Then every functional $f \in P'^{**}$ has the form

$$f(x) = \bar{f}_0(xy), \quad (*)$$

where y is some element of R , and such a representation is unique. The converse is also true: for every $y \in R$, the functional f defined by formula $(*)$ belongs to P' . Thus, in its composition, R may be regarded as the space conjugate to P .

Theorem 5. In the space $L[0, 1]$ there exists a fundament R which contains all $L_p[0, 1]$ for $p > 1$ and which, for a certain choice of norm, is a reflexive KB -space.

* A linear subspace $Y \subset X$ is called a fundament if it is contained in X normally and completely ⁽²⁾, p. 112

** If X is a normed space, then by X' we denote the space conjugate to X .

Let us give a brief outline of the proof of this theorem. Let P be the set of all functions from $L[0, 1]$ for which

$$\|x\| = \left[\sum_{p=1}^{\infty} \frac{1}{p^2} \|x^2\|_{L_p} \right]^{1/2} < +\infty.$$

With the aid of Theorem 1 one can verify that P is a reflexive KB -space. Now we use the preceding assertion on the general form of a linear functional, taking

$$f_0(x) = \int_0^1 x(t) dt,$$

where $x \in P$. In this case Q coincides with L . Then the space R will consist of all such $y \in L$ that

$$\int_0^1 |x(t)y(t)| dt < +\infty$$

for every $x \in P$. This R is the required one, if it is endowed with the topology of the space conjugate to P .

Remark 1. The constructed space R , although it is a reflexive KB -space, is not a space of type X_p for any choice of the KB -space X and number $p > 1$.

Remark 2. D. A. Vladimirov drew my attention to the fact that R is not an Orlicz space, but contains all reflexive Orlicz spaces. Moreover, if x is an arbitrary element of R , then all functions equimeasurable with x are also contained in R .

Theorem 6. Let X be a K -space, and let X_1 and X_2 be two of its fundamentals that are reflexive KB -spaces with norms $\| \cdot \|_1$ and $\| \cdot \|_2$, respectively. Put $X_3 = X_1 \cap X_2$. Introduce on X_3 the norm

$$\|x\|_3 = \max\{\|x\|_1, \|x\|_2\}.$$

Then X_3 is a reflexive KB -space.

Let us note that from this theorem there follows an answer to one of Halperin's questions ((4), pp. 247–249). Namely, using Halperin's notation, let us indicate that the space L^λ , constructed from the function

$$\lambda(u) = \lambda_{(\omega_1, \omega_2)}^p(u) = \max(\lambda_{(\omega_1)}^p(u), \lambda_{(\omega_2)}^p(u)),$$

is reflexive, if the spaces $L_{(\omega_1)}^p$ and $L_{(\omega_2)}^p$ are reflexive.

Theorem 7. Under the conditions of Theorem 6, on the set $X_1 + X_2$, i.e. in the linear hull of X_1 and X_2 , one can define a norm in such a way that it will be a reflexive KB -space.

It follows from Theorems 6 and 7 that all fundamentals of a given K -space X that are reflexive KB -spaces form a distributive structure if ordered by inclusion. Denote this structure by \mathfrak{S} . If X contains a unit e , put

$$\mathfrak{T} = \{Y : e \in Y \in \mathfrak{S}\}.$$

In this case \mathfrak{T} is a substructure of \mathfrak{S} .

Let, in particular, X be a KB -space with an additive norm and a unit. Take the functional

$$f_0(x) = \|x_+\| - \|x_-\|$$

and use the remark made earlier. Then to every reflexive KB -space Y that is a fundamental in X and contains the unit (that is, $Y \in \mathfrak{T}$), its conjugate Y' is assigned uniquely, and $Y' \in \mathfrak{T}$. One can prove that in the distributive structure \mathfrak{T} there are

the following relations between conjugate spaces hold; namely, for any $Y_1, Y_2 \in \mathfrak{S}$,

$$(Y_1 \vee Y_2)' = Y_1' \wedge Y_2', \quad (Y_1 \wedge Y_2)' = Y_1' \vee Y_2'.$$

If $Y_1 = Y_1'$, then Y_1 is a Hilbert space X_2 .

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Note: Figure translations are in progress. See original paper for figures.

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