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Abstract

Full Text

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On the Properties of the Limit Function of a Sequence of Linear Aggregates

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A. F. Leont' ev posed the following problem ([1]). Suppose that $f(z)$ is an entire function and λ_k are complex numbers such that the system $\{f(\lambda_k z)\}$ is not complete in the disk $|z - z_0| < R$. Suppose further that the sequence of linear aggregates

$$\mathcal{P}_n(z) = \sum_{k=1}^{q_n} c_{k,n} f(\lambda_k z), \qquad n = 1, 2, \dots, \tag{1}$$

converges uniformly in the disk $|z - z_0| \le R$. It is required to determine what specific properties the limit function of this sequence possesses.

In the works of Pólya, Valiron, and Leont' ev ([2–4]), the properties of the limit function of the sequence (1) were studied in the simplest, but perhaps the most important, special case, when $f(z) = e^z$. In ([5]) it is obtained as a consequence of a more general theorem that if $f(z)$ has the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = 1 + \sum_{n=1}^{\infty} \frac{z^n}{\psi(1) \cdots \psi(n)}, \tag{2}$$

$\psi(x) = \sum_{l=1}^{p} a_l x^l$, $p > 1$, $\psi(k) \neq 0$, $k = 1, 2, \dots$, the numbers λ_n satisfy the condition $\lim_{n\to\infty} \frac{n^p}{|\lambda_n|} = 0$, and the sequence (1) converges uniformly in some domain, then the limit function is single-valued everywhere and its domain of existence is simply connected.

Using the theory of generalized derivatives in the sense of A. O. Gel' fond and A. F. Leont' ev ([6]), one can obtain a fairly broad generalization of the last result.

Suppose that the entire function $f(z)$ has the form (2), where $\psi(z)$ is an entire function of growth not exceeding first order of minimal type such that

$$\psi(0) = 0, \qquad \psi(n) \neq 0, \qquad n = 1, 2, \dots, \tag{3}$$

$$\lim_{n\to\infty} |\psi(1) \cdots \psi(n)|^{1/n} = \infty. \tag{4}$$

Condition (4) will be satisfied, for example, if $|\psi(x)| \to \infty$ as x increases without bound along the positive real axis.

Let $\{\lambda_n\}$ be a sequence of pairwise distinct and nonzero complex numbers such that $\sum_{n=1}^{\infty} \frac{\alpha_n}{|\lambda_n|} < \infty$, where α_n are natural numbers, $\alpha_n \geq 1$, $n = 1, 2, \ldots$. Denote by $\omega(z)$ the function $\sum_{n=1}^{\infty} \psi(n) z^{n-1}$; po

by the Wigert-Löw theorem, $\omega(z)$ is an entire function of $\dfrac{1}{1-z}$. Further, let

$$\nu(R) = \max_{|1-z|=1/R} |\omega(z)|,$$

and let $\mu(R)$ be the function inverse to $\nu(R)$.

Introduce the functions

$$\mathcal{L}(x) = x^{p_1} \prod_{k=1}^{\infty} \left(1 + \frac{x}{\lambda_k}\right)^{\alpha_k} = \sum_{k=p_1}^{\infty} c_k x^k,$$

where p_1 is a natural number, $p_1 \geq 0$, and

$$\widetilde{\mathcal{L}}(x) = x^{p_1} \prod_{k=1}^{\infty} \left(1 + \frac{x}{|\lambda_k|}\right)^{\alpha_k} = x^{p_1} \prod_{k=1}^{\infty} \left(1 + \frac{x}{\mu_k}\right) = \sum_{k=p_1}^{\infty} \sigma_k x^k.$$

Suppose that the numbers λ_n satisfy the condition

$$\lim_{n\to\infty} \frac{n}{\mu\left(n/\sqrt[n]{\sigma_n}\right)} = \tau < \infty. \tag{5}$$

Consider the sequence

$$\mathfrak{P}_n(z) = \sum_{k=1}^{l_n} \sum_{j=0}^{\alpha_k - 1} z^j f^{(j)}(z\lambda_k) c_{k,j}^{(n)}, \qquad n = 1, 2, \ldots, \tag{6}$$

where l_n are natural numbers.

Denote by $d(\tau, z)$ the function defined as follows:

1) for a transcendental entire function $\psi(z)$ of class $[1, 0]$,

$$d(\tau, z) = \begin{cases} (e^\tau - 1)|z|, & \text{if } \tau < \ln 2, \\ e^\tau |z|, & \text{if } \tau \geq \ln 2; \end{cases}$$

2) for

$$\psi(z) = \sum_{l=1}^{p} \alpha_l z^l, \qquad p > 1,$$

$$d(\tau, z) = \begin{cases} (\Delta)^p, & \text{if } (\Delta)^p > |z|, \\ \Delta |z|^{1-1/p}, & \text{if } (\Delta)^p \le |z|, \end{cases}$$

where

$$(\Delta)^p = \frac{\tau}{p!} \left(\frac{2p^p \tau}{(p-1)^{p-1} e} \right)^p;$$

3) for $\psi(z) \equiv z$,

$$d(\tau, z) = \frac{\tau^2}{e}.$$

Theorem 1. Let condition (5) be satisfied, and let the sequence (6) converge uniformly in some disk

$$|z - z_0| \le d(\tau, z_0) + h.$$

Then the limits

$$\lim_{n\to\infty} c_{k,j}^{(n)} = c_{k,j}, \qquad 0 \le j \le \alpha_k - 1, \quad k = 1, 2, \dots$$

exist.

Theorem 2. The sequence (6) and the sequence

$$\widetilde{\mathfrak{P}}_n(z) = \sum_{k=1}^{\tilde{l}_n} \sum_{j=0}^{\alpha_k - 1} z^j f^{(j)}(z\lambda_k) \tilde{c}_{k,j}^{(n)}, \qquad n = 1, 2, \dots,$$

satisfying the conditions of Theorem 1, converge to one and the same limiting function if and only if the limits of the corresponding coefficients are equal to one another:

$$c_{m,j} = \tilde{c}_{m,j}, \qquad j = 0, 1, \dots, \alpha_m - 1; \quad m = 1, 2, \dots.$$

Theorem 3. Let the sequence (6) converge uniformly in the disk

$$|z - z_0| \le d(\tau, z_0) + h, \qquad h > 0,$$

to a function $P(z)$, and let $D_\tau^{z_0}$ be the connected set on the Riemann surface of the function $P(z)$ containing z_0 and all points z_1 such that $P(z)$ is analytic in the disk

$$|z - z_1| \leq d(\tau, z_1).$$

Then uniformly inside $D_\tau^{z_0}$,

$$P(z) = \lim_{m\to\infty} \sum_{k=1}^{m} \sum_{j=0}^{\alpha_k-1} f^{(j)}(z\lambda_k) z^j \sum_{s=j}^{\alpha_k-1} c_{k,s} c_s^j \mathcal{L}_{m+1,\infty}^{(s-j)}(\lambda_k),$$

where

$$c_{k,s} = \lim_{n\to\infty} c_{k,s}^{(n)}, \qquad \mathcal{L}_{m+1,\infty}(x) = \prod_{k=m+1}^{\infty} \left(1 - \frac{x}{\lambda_k}\right)^{\alpha_k}.$$

In this case the domain $D_\tau^{z_0}$ is simply connected and one-connected.

Corollary. If the numbers λ_n are such that

$$\lim_{n\to\infty} \frac{n}{\mu\left(n! \sqrt[n]{\sigma_n}\right)} = 0$$

and the sequence (6) converges uniformly in some circle, then the domain of existence of the limiting function is simply connected and one-connected.

One can also obtain characteristics of certain boundary properties of the domain $D_\tau^{z_0}$. Suppose that for all z_1 and z_2,

$$|d(\tau, z_1) - d(\tau, z_2)| < |z_1 - z_2|.$$

Next, let α be any boundary point of $D_\tau^{z_0}$, and let β_α be a singular point of $P(z)$ lying on the circle $|\alpha - t| = d(\tau, \alpha)$; let $\rho(\beta_\alpha)$ be the distance from β_α to the next nearest singular point of $P(z)$. The quantity $\rho(\beta_\alpha)$ can be estimated from above.

Theorem 4. Under the hypotheses of Theorem 3, the inequality

$$\rho(\beta_\alpha) \leq 2d(\tau, |\beta_\alpha| + \rho(\beta_\alpha)/2)$$

holds. In addition, if the function $d(\tau, z)$ is such that

$$|d(\tau, z_1) - d(\tau, z_2)| \leq K|z_1 - z_2|, \qquad K < 1,$$

where z_1, z_2 are arbitrary complex numbers, then

$$\rho(\beta_\alpha) \leq \frac{2}{1-K}\, d(\tau, \beta_\alpha).$$

Under certain conditions the limiting function of the sequence (6) can be represented in the form of a series. For simplicity of formulation we restrict ourselves to the case when $p_1 = 0$, $\alpha_k = 1$, $k = 1, 2, \ldots$, and denote by α-quantities the limits

$$\lim_{n\to\infty} c_{s,n},$$

which exist under the assumptions of Theorem 1.

Theorem 5. Suppose that $f(z)$ is an entire function of the form (2), where $\psi(z)$ is an entire function of class $[1, 0]$ satisfying condition (3).

Suppose, furthermore, that the numbers λ_n are such that condition (5) is satisfied and one of the two conditions

$$\text{a)} \quad \overline{\lim_{n\to\infty}} \frac{\left|\ln \dfrac{1}{|\mathcal{L}'(\lambda_n)|}\right|}{\ln|\lambda_n|} < \infty; \qquad \text{b)} \quad \overline{\lim_{n\to\infty}} \frac{\nu\left(\left|\ln \dfrac{1}{|\mathcal{L}'(\lambda_n)|}\right|\right)}{|\lambda_n| \left|\ln \dfrac{1}{|\mathcal{L}'(\lambda_n)|}\right|} < \infty.$$

Finally, suppose that the sequence (1) converges uniformly in the circle

$$|z - z_0| \leq d(\tau, z_0) + h, \qquad h > 0.$$

Then in the domain $D_\tau^{z_0}$

$$\mathcal{P}(z) = \sum_{k=1}^{\infty} \alpha_k f(\lambda_k z), \tag{7}$$

and the series (7) converges uniformly inside $D_\tau^{z_0}$. If the domain $D_\tau^{z_0}$ contains at least one circle

$$|z - z_1| \leq d(\tau, z_1),$$

then the expansion of $\mathcal{P}(z)$ into a series of the form (7) is unique.

Under various concrete assumptions concerning the growth of the function $\psi(z)$ of class $[1, 0]$, one can obtain a number of corollaries from Theorems 1-5.*

1. In the particular case when

$$\psi(x) = \sum_{l=1}^{p} \alpha_l x^l, \qquad p > 1, \qquad \psi(n) \neq 0, \quad n = 1, 2, \dots,$$

Theorems 1-4 were obtained earlier by A. F. Leont' ev.

2. $\psi(z)$ is a transcendental entire function of growth not exceeding order $\rho < 1$ and of finite type σ. If the numbers λ_n are such that

$$\lim_{n\to\infty} \frac{|s_n|^{\rho/(1-\rho)}}{\ln|\lambda_n|} = c < \infty,$$

where

$$s_n = \sum_{k=1}^{n} \alpha_k,$$

then condition (5) is satisfied and

$$\tau = (c)^{(1-\rho)/\rho} \rho(\sigma)^{1/\rho}.$$

In this

* After the article had been submitted for publication, I learned that the sequence (1) in the case (2), when $\psi \in [1, 0]$, had been considered by I. F. Lokhin, but his results were not published.

in that case Theorems 1-5 are valid; if, in particular, $\tau < \ln 2$, then for any boundary point α of the domain D_0^z

$$\rho(\beta_\alpha) \leq \frac{2(e^\tau - 1)}{(2 - e^\tau)} |\beta_\alpha|.$$

3. $\psi(z)$ is a transcendental entire function of order not exceeding ρ, $\rho < 1$. If

$$\overline{\lim_{n\to\infty}} \frac{\ln s_n}{\ln\ln|\lambda_n|} < \frac{1-\rho}{\rho},$$

then condition (5) is satisfied, and moreover $\tau = 0$. From Theorems 1-4 in this case it follows, in particular, that if the sequence (6) converges uniformly in some circle $|z - z_0| \leq h$, then the Riemann surface of the limiting function is simply connected and single-sheeted. If $\alpha_n = 1$, $n = 1, 2, \dots$,

$$\overline{\lim_{n\to\infty}} \frac{\ln n}{\ln\ln|\lambda_n|} < \frac{1-\rho}{\rho},$$

and if, in addition, one of the following two conditions is satisfied:

a)

$$\overline{\lim_{n\to\infty}} \frac{\ln\left|\dfrac{1}{\mathcal{L}'(\lambda_n)}\right|}{\ln|\lambda_n|} < \infty, \qquad \text{if } \rho \geq \frac{1}{2},$$

or

b)

$$\overline{\lim_{n\to\infty}} \frac{\ln\left|\ln\dfrac{1}{|\mathcal{L}'(\lambda_n)|}\right|}{\ln\ln|\lambda_n|} < \frac{1-\rho}{\rho}, \qquad \text{if } \rho < \frac{1}{2},$$

then the limiting function can be represented in the form of the series (7), converging uniformly inside its entire domain of existence.

4. If $\psi(z)$ is a transcendental entire function of zero order such that

$$\lim_{r\to\infty} \frac{\ln M_r(\psi)}{(\ln r)^\alpha} = A < \infty, \quad \alpha < 1, \quad M_r(\psi) = \max_{|x|=r} |\psi(x)|,$$

and the numbers λ_n are such that

$$\overline{\lim_{n\to\infty}} \frac{(\ln s_n)^\alpha}{\ln|\lambda_n|} < \frac{1}{A},$$

then from the uniform convergence of the sequence (6) in some circle there follows single-sheetedness and simple connectedness of the domain of existence of the limiting function. If $\alpha_n = 1$, $n = 1, 2, \ldots$,

$$\overline{\lim_{n\to\infty}} \frac{(\ln n)^\alpha}{\ln|\lambda_n|} < \frac{1}{A}$$

and if

$$\overline{\lim_{n\to\infty}} \frac{\ln\left|\ln\dfrac{1}{|\mathcal{L}'(\lambda_n)|}\right|}{\ln|\lambda_n|} < \infty,$$

then the limiting function can be represented by the series (7), converging uniformly inside its domain of existence.

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Note: Figure translations are in progress. See original paper for figures.