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Abstract

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ON GENERALIZED RESOLVENTS OF AN ISOMETRIC OPERATOR

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1. The development of the spectral theory of symmetric* non-self-adjoint operators in a Hilbert space H was initiated by Carleman ([1]) and Stone ([10]). M. A. Naimark ([6]) was the first to give a general definition of the spectral function of an arbitrary symmetric operator and established an important theorem according to which a family of operators $E_t(-\infty < t < +\infty)$ in H is the spectral function of a symmetric operator A in H if and only if it admits the representation

$$E_t h = P\widetilde{E}_t h \quad (h \in H), \tag{1}$$

where $\widetilde{E}_t(-\infty < t < +\infty)$ is the spectral function of some self-adjoint extension $\widetilde{A}$ of the operator A in a Hilbert space $\widetilde{H} \supset H$, and P is the operator of orthogonal projection in $\widetilde{H}$ onto H. The definition of M. A. Naimark and formula (1) admit a generalization to any closed Hermitian operator ([12]). Relation (1) may be taken as the basis for the definition of the spectral function of a closed Hermitian operator.

A formula for the generalized resolvents of an arbitrary symmetric operator, convenient for the actual construction of spectral functions, was obtained by A. V. Shtraus ([11,12]). In another form, generalized resolvents of symmetric operators with defect index $(1, 1)$ and (m, m) $(m < \infty)$ had earlier been described by M. A. Naimark ([7]) and M. G. Krein ([3,4]). A. V. Shtraus ([12]) also studied the properties characterizing generalized resolvents of Hermitian operators and proved an important theorem on the characteristic properties of generalized resolvents of a closed Hermitian operator.

In the present note we consider properties characterizing generalized resolvents of isometric operators, and establish that, by means of a formula analogous to the formula of A. V. Shtraus, all generalized resolvents of a closed isometric operator can be specified. In doing so, we largely follow the method of ([12]). We note that the spectral functions of isometric operators in the special case when the operator is defined on all of H were considered by A. I. Plesner ([9]).

2. Let U be a closed isometric operator acting in a Hilbert space H, and let $\widetilde{U}$ be an arbitrary unitary extension of the operator U (such an extension always exists, possibly with passage to an enlarged space $\widetilde{H} \supset H$). In what follows we shall denote by $\widetilde{E}_t$ $(0 \le t \le 2\pi)$ the spectral function of the operator $\widetilde{U}$, by P the operator of orthogonal projection in $\widetilde{H}$ onto H, and by D_B and Δ_B, respectively, the domain of definition and the range of values of any operator B.

Definition 1. A family of operators E_t $(0 \le t \le 2\pi)$, acting in the Hilbert space H, is called a **spectral function** of the operator U if, for every $h \in H$,

$$E_t h = P\widetilde{E}_t h.$$

* Recall that a linear operator A, acting in a Hilbert space H, is called **Hermitian** if, for any elements f, g of its domain of definition D_A, $(Af, g) = (f, Ag)$. If, in addition, the manifold D_A is dense in H, then the operator A is called **symmetric**.

Definition 2. A family of operators R_ζ $(|\zeta| \neq 1)$ in H is called a **generalized resolvent** of the operator U if

$$R_\zeta h = P(E - \zeta\widetilde{U})^{-1} h \qquad (h \in H). \tag{2}$$

A given isometric operator has, generally speaking, an infinite set of spectral functions and the corresponding generalized resolvents. If E_t, R_ζ, and $\widetilde{U}$ are related as indicated in the definitions, then we shall say that E_t and R_ζ are **generated by the extension** $\widetilde{U}$.

Theorem 1. In order that a family of linear bounded operators R_ζ, acting in H $(D_{R_\zeta} = H)$ and depending on the complex parameter ζ $(|\zeta| \neq 1)$, be the generalized resolvent of some closed isometric operator, it is necessary and sufficient that the following conditions be satisfied:

1) for some $\zeta_0 \neq 0$ $(|\zeta_0| < 1)$ there exists a subspace $L \subset H$ such that, for every ζ $(|\zeta| \neq 1)$ and every $f \in L$, the equality

$$(\zeta R_\zeta - \zeta_0 R_{\zeta_0}) f = (\zeta - \zeta_0) R_\zeta R_{\zeta_0} f;$$

holds;

2) for every $\varphi \in H \ominus \overline{R_{\zeta_0} L}$, $R_0 \varphi = \varphi$;

3) for every ζ $(|\zeta| < 1)$ and every $h \in H$ the inequality

$$\mathrm{Re}(R_\zeta h, h) \ge \frac{1}{2}(h, h);$$

holds;

4) for every $h \in H$, $R_\zeta h$ is an analytic vector function of ζ in the unit disk $|\zeta| < 1$;

5) for every ζ $(0 < |\zeta| < 1)$

$$R_\zeta^* = E - R_{1/\bar{\zeta}}.$$

In Theorem 1 the question concerned intrinsic properties characterizing the generalized resolvents of a closed isometric operator. At the same time, however, the necessary and sufficient conditions were not clarified under which the family of operators R_ζ would serve as the generalized resolvent of a given closed isometric operator. The following theorem gives an answer to this question.

Theorem 2. In order that a family of linear bounded operators R_ζ in H $(D_{R_\zeta} = H,\ |\zeta| \neq 1)$ serve as the generalized resolvent of a closed isometric operator U in H, it is necessary and sufficient that conditions 3)–5) of Theorem 1 be satisfied, as well as the conditions:

1′) for every ζ $(|\zeta| \neq 1)$ and every $g \in D_U$ the equality

$$R_\zeta(E - \zeta U)g = g;$$

holds;

2′) for every $\varphi \in H \ominus \overline{D_U}$, $R_0\varphi = \varphi$.

We now formulate the theorem on the general formula for generalized resolvents of a closed isometric operator, which can be proved on the basis of Theorem 2. In view of the relation

$$R_{1/\bar{\zeta}} = E - R_\zeta^*$$

it is sufficient to establish this formula for all values of ζ with modulus less than one.

Theorem 3. Every generalized resolvent R_ζ of a closed isometric operator U in H is representable in the form:

$$R_\zeta = [E - \zeta(U \oplus \Phi(\zeta))]^{-1} \qquad (|\zeta| < 1), \tag{3}$$

where $\Phi(\zeta)$ is some linear operator from $H \ominus D_U$ into $H \ominus \Delta_U$, which is an operator-valued function analytic in the unit disk and not exceeding one in norm. Conversely, every operator-valued function $\Phi(\zeta)$ possessing the listed properties determines by formula (3) some generalized resolvent of the operator U.

Remark 1. From formula (3) it follows that different $\Phi(\xi)$ correspond to different generalized resolvents R_ξ.

Remark 2. With the aid of formula (3) one easily obtains the result of B. Sz.-Nagy ([8]):

If T is a contraction of a Hilbert space H, then in some space $\widetilde{H} \supset H$ there exists a unitary operator $\widetilde{U}$ such that

$$T^n h = P\widetilde{U}^n h \qquad (h \in H;\ n = 0, 1, 2, \ldots). \tag{4}$$

Indeed, the operator T can be represented in the form $T = U \oplus \Phi$, where U is a closed isometric operator and $\|\Phi\| \leq 1$ (for example, as U one may take the operator with $D_U = \{0\}$). Therefore, by virtue of formulas (2) and (3), we have

$$(E - \xi T)^{-1} h = P(E - \xi\widetilde{U})^{-1} h \qquad (h \in H,\ |\xi| < 1),$$

whence (4) follows.

Theorem 4. *Let U be a closed isometric operator which has no nonzero fixed elements. Then formula (3), where $\Phi(\xi)$ satisfies the same conditions as in Theorem 3, determines a generalized resolvent of the operator U, generated by some unitary extension $\widetilde{U}$ having no nonzero fixed elements, if and only if, for every $\varphi \in H \ominus \overline{\Delta}_{E-U}$, the condition*

$$\lim_{\xi \to 1} (1 - \xi)([E - \xi(U \oplus \Phi(\xi))]^{-1} \varphi, \varphi) = 0$$

$$\left(-\frac{\pi}{2} + \varepsilon < \arg(1 - \xi) < \frac{\pi}{2} - \varepsilon; \quad 0 < \varepsilon < \frac{\pi}{2}\right). \tag{5}$$

is fulfilled.

Let the operator U be the same as in Theorem 4. Denote by P_1 and P_2 the operators of orthogonal projection in H respectively onto $H \ominus D_U$ and $H \ominus \Delta_U$. On the set of elements $P_1(H \ominus \overline{\Delta}_{E-U})$ define an operator V by the formula

$$V P_1 h = P_2 h \qquad (h \in H \ominus \overline{\Delta}_{E-U}).$$

The operator V is isometric ([2]).

Lemma. *Let the operator U be the same as in Theorem 4, and let $\Phi(\xi)$ be the same as in Theorem 3. If for every $\varphi \in H \ominus \overline{\Delta}_{E-U}$ relation (5) is fulfilled, then $\Phi(\xi)\psi = V\psi$ only for $\psi = 0$.*

Using Theorem 4 and the lemma, one can prove the following proposition.

Theorem 5. *Every generalized resolvent R'_λ of a closed Hermitian operator A in H is representable in the form*

$$R'_\lambda = (A_{F(\lambda)} - \lambda E)^{-1} \qquad (\operatorname{Im}\lambda > 0), \tag{6}$$

where $F(\lambda)$ is some linear operator from $\mathfrak{N}_i = H \ominus (A - iE)D_A$ into $\mathfrak{N}_{-i} = H \ominus (A + iE)D_A$, satisfying the conditions:

1) $\|F(\lambda)\| \leq 1$;
2) $F(\lambda)$ is an operator-valued function of the parameter λ, analytic in the upper half-plane;
3) for every $\varphi \in H \ominus \overline{D}_A$

$$\lim_{\lambda\to\infty} \lambda\big((A_{F(\lambda)} - \lambda E)^{-1}\varphi, \varphi\big) = (-\varphi, \varphi) \qquad \left(\varepsilon < \arg\lambda < \pi - \varepsilon;\ 0 < \varepsilon < \frac{\pi}{2}\right),$$

and the operator $A_{F(\lambda)}$ is defined by the formula

$$A_{F(\lambda)} g = Af + iF(\lambda)\psi_i + i\psi_i$$

on the manifold of elements of the form

$$g = f + F(\lambda)\psi_i - \psi_i \qquad (f \in D_A,\ \psi_i \in \mathfrak{N}_i).$$

Conversely, by every operator function possessing properties 1)-3), a certain generalized resolvent of the operator A is determined by formula (6).

This proposition generalizes Theorem 7 of the work [12].

The result contained in Theorem (5) was previously established by another method by B. I. Loshkarev [5]. Let us note that condition 3, which appears in the theorem of the work [5], turns out to be superfluous.

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Note: Figure translations are in progress. See original paper for figures.