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Abstract

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CONJUGATE FUNCTIONS OF TWO VARIABLES AND DOUBLE CONJUGATE TRIGONOMETRIC SERIES

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1. Let $f(x) \in L(-\pi, \pi)$ be a 2π-periodic function. As is known (see ([11]), p. 528), there exists almost everywhere the function

$$\bar{f}(x) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \operatorname{ctg} \frac{t}{2} \, dt,$$

which is called the conjugate function to $f(x)$. It is also well known (see ([10]), p. 227) that for a summable function $f(x)$ the function $\bar{f}(x)$ need not be summable. However, as A. Zygmund showed ([2]), if

$$|f(x)| \log^+ |f(x)| \in L(-\pi, \pi),$$

then $\bar{f}(x)$ is summable. Let us note that the summability of the function

$$|f(x)| \log^+ |f(x)|$$

is essential for the integrability of the function $\bar{f}(x)$ and cannot, in general, be replaced by any weaker condition. This follows from the following theorem of M. Riesz ([1]):

If $f(x) \in L(-\pi, \pi)$, $f(x) \geq 0$ and the function $\bar{f}(x)$ is summable, then

$$f(x) \log^+ f(x) \in L(-\pi, \pi).$$

Furthermore, if $f(x)$ is bounded, then $\bar{f}(x)$ may fail to be bounded.

However, as M. Kinukawa ([8]) and R. Turán ([6]) showed, if $f(x)$ is even and bounded, then the functions

$$\frac{1}{x} \int_0^x \bar{f}(t) \, dt, \qquad \int_x^{\pi} \frac{\bar{f}(t)}{t} \, dt$$

are bounded.

Let us now consider a function of two variables $f(x, y)$. Suppose that it is periodic with respect to each of the variables and $f(x, y) \in L(R)$, where

$$R = [-\pi, \pi;\ -\pi, \pi].$$

Let the series

$$\sum_{m,n=0}^{\infty} \lambda_{mn} A_{mn}(x, y) \tag{1}$$

be the double Fourier-Lebesgue series of the function $f(x, y)$, where $\lambda_{00} = 1/4$, $\lambda_{0n} = \lambda_{m0} = 1/2$ for $m, n > 0$; $\lambda_{mn} = 1$ for $m, n > 0$, and

$$A_{mn}(x, y) = a_{mn} \cos mx \cos ny + b_{mn} \sin mx \cos ny +$$
$$+ c_{mn} \cos mx \sin ny + d_{mn} \sin mx \sin ny.$$

Let us now consider the conjugate trigonometric series

$$\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \lambda_{mn}(-b_{mn} \cos mx \cos ny + a_{mn} \sin mx \cos ny -$$
$$- d_{mn} \cos mx \sin ny + c_{mn} \sin mx \sin ny), \tag{2}$$

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \lambda_{mn}(-c_{mn} \cos mx \cos ny - d_{mn} \sin mx \cos ny +$$
$$+ b_{mn} \cos mx \sin ny + a_{mn} \sin mx \sin ny), \tag{3}$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (d_{mn} \cos mx \cos ny - c_{mn} \sin mx \cos ny - b_{mn} \cos mx \sin ny + a_{mn} \sin mx \sin ny). \tag{4}$$

Under certain conditions the series (2), (3), (4) are summable by Abel' s method respectively to the functions

$$\bar{f}_1(x, y) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + s, y) \operatorname{ctg} \frac{s}{2}\, ds,$$

$$\bar{f}_2(x, y) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, y + t) \operatorname{ctg} \frac{t}{2}\, dt,$$

$$\bar{f}(x,y)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+s,y+t)\operatorname{ctg}\frac{s}{2}\operatorname{ctg}\frac{t}{2}\,ds\,dt.$$

A. Zygmund ([5]) proved that if $|f(x,y)|\log^+|f(x,y)|\in L(R)$, then the function $\bar{f}(x,y)$ exists almost everywhere, and

$$\bar{f}(x,y)=\frac{1}{4\pi^2}\lim_{(\varepsilon,\eta)\to 0}\int_{\varepsilon}^{\pi}\int_{\eta}^{\pi}\frac{f(x+s,y+t)-f(x-s,y+t)-f(x+s,y-t)+f(x-s,y-t)}{\operatorname{tg}\frac{s}{2}\ \operatorname{tg}\frac{t}{2}}\,ds\,dt.$$

Further, there exists an example of a function of two variables for which $f(x,y)\in L(R)$, but $\bar{f}(x,y)$ does not exist (see ([9], [12])) almost everywhere even as $(\varepsilon,\varepsilon)\to 0$. It is known ([13]) that if $|f(x,y)|[\log^+|f(x,y)|]^2\in L(R)$, then $\bar{f}(x,y)\in L(R)$. The question is asked: if $f(x,y)\in L(R)$, $f(x,y)\geq 0$, and the function $\bar{f}(x,y)$ is summable, can one assert that $f(x,y)[\log^+ f(x,y)]^2\in L(R)$? It is known ([4]) that if $f(x,y)\in L(R)$, then almost everywhere

$$\lim_{(m,n)\to\infty}\sigma_{mn}(x,y)=f(x,y),$$

where $\sigma_{mn}(x,y)$ are the Cesàro means of the series (1). However, how the Cesàro means of the series (2), (3), (4) behave for an arbitrary summable function $f(x,y)$ is still unknown.

Finally, the question is posed: are the theorems of M. Kinukawa ([8]) and R. Turán ([6]) valid for functions of two variables?

2. In the present article we give results which answer the questions posed above. In addition, we consider some other questions related to those indicated.

Theorem 1. *There exists a nonnegative 2π-periodic function $f(x,y)$ such that $f(x,y)[\log^+ f(x,y)]^\alpha\in L(R)$ for all $\alpha\in[0,1)$ and*

$$\lim_{(\varepsilon_m,\eta_n)\lambda\to 0}\left|\int_{\varepsilon_m}^{\pi}\int_{\eta_n}^{\pi}\frac{f(x+s,y+t)-f(x-s,y+t)-f(x+s,y-t)+f(x-s,y+t)}{\operatorname{tg}s/2\ \operatorname{tg}t/2}\times ds\,dt\right|=+\infty$$

on a set of positive measure for any $\varepsilon_m\to 0$, $\eta_n\to 0$ and for any $\lambda\geq 1$.

In particular, it follows from this theorem that the results of I. Vatanabe ([7]) are erroneous.

Theorem 2. *There exists a nonnegative 2π-periodic function $f(x,y)\in L(R)$ such that $\bar{f}(x,y)\in L(R)$, but $f(x,y)[\log^+ f(x,y)]^\alpha\notin L(R)$ for any $\alpha\in(0,\beta]$, $\beta>0$.*

It follows from this theorem that the theorem of M. Riesz ([1]) is false for conjugate functions of several variables.

Theorem 3. There exists a nonnegative 2π-periodic function $f(x, y)$ such that

$$\bar{f}(x, y) \log^+ \bar{f}(x, y) \in L(R), \qquad \bar{f}(x, y) \in L(R),$$

but

$$f(x, y)\,[\log^+ f(x, y)]^{1+\varepsilon} \notin L(R)$$

for all $\varepsilon \in (0, \beta]$.

We shall now denote by $\bar{\sigma}^{(1)}_{mn}(x, y)$, $\bar{\sigma}^{(2)}_{mn}(x, y)$, $\bar{\sigma}^{(3)}_{mn}(x, y)$ the Cesàro means, respectively, of the series (2), (3), and (4). The following is true.

Theorem 4. There exists a 2π-periodic function $f(x, y)$ such that

$$|f(x, y)|\,[\log^+ |f(x, y)|]^{\alpha} \in L(R)$$

for all $\alpha \in [0, 1)$, but

$$\overline{\lim_{(m,n)_\lambda \to \infty}} \left|\bar{\sigma}^{(i)}_{mn}(x, y)\right| = +\infty \qquad (i = 1, 2, 3)$$

on a set of positive measure.

We note that the theorem is valid for the Abel and $(C; \alpha, \beta)$ methods, and the assertions cease to hold if

$$|f(x, y)|\,[\log^+ |f(x, y)|]^{\alpha} \in L(R)$$

for some $\alpha \geq 1$. The results obtained show that the corresponding assertions of Marcinkiewicz and Zygmund ([3]), generally speaking, are also false for multiple trigonometric series.

Theorem 5. There exists a 2π-periodic function $f(x, y) \in L(R)$ for which the series (1) and (2) diverge everywhere, but the series (3) and (4) are uniformly $(C; \alpha, \beta)$-summable for any $\alpha > 1$, $\beta > -1$.

On the basis of Theorem 4 one obtains

Theorem 6. There exists an analytic function of two variables, defined in the bicylinder $|z| < 1$, $|\eta| < 1$ and representable by a double integral of Cauchy type

$$f(z, \eta) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s, t) \frac{\tau + z}{\tau - z} \frac{\sigma + \eta}{\sigma - \eta}\, ds\, dt,$$

which, on a set of positive planar measure, has no boundary values when $(r, \rho)_\lambda \to 1$, while

$$|f(x, y)|\,[\log^+ |f(x, y)|]^{\alpha} \in L(R)$$

for all $\alpha \in [0, 1)$, $z = \rho e^{ix}$, $\eta = r e^{iy}$, $\tau = e^{is}$, $\sigma = e^{it}$.

The theorem ceases to hold if

$$|f(x,y)|\,[\log^+|f(x,y)|]^\alpha \in L(R)$$

for some $\alpha \geq 1$. As for the results of M. Kinukawa ([8]) and P. Turan ([6]), they are likewise false for functions of two variables, since the following is true.

Theorem 7. There exists an even bounded 2π-periodic function $f(x,y)$ for which the functions

$$\frac{1}{x}\int_0^x \bar{f}_i(s,y)\,ds, \qquad \int_x^\pi \frac{\bar{f}_i(s,y)}{s}\,ds \qquad (i=1,2)$$

are unbounded.

It is possible, by an example, to show the unboundedness of the functions

$$\frac{1}{xy}\int_0^x\int_0^y \bar{f}_i(s,t)\,ds\,dt, \qquad \frac{1}{xy}\int_0^x\int_0^y \bar{f}(s,t)\,ds\,dt \qquad (i=1,2).$$

Finally, we note that results analogous to those presented in this note are also valid for n-fold trigonometric series.

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Note: Figure translations are in progress. See original paper for figures.