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Abstract

Full Text

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τ-POLYNOMIALS, τ-ANALYTIC FUNCTIONALS

(Presented by Academician P. S. Aleksandrov on 23 I 1964)

In ([1]) the concept of τ-differentiability was introduced and the general form was found of a functional whose $(n+1)$-st τ-derivative is identically zero. Such functionals were called τ-polynomials. Here we shall consider some of their properties; in particular, we shall obtain for them an expansion analogous to the Taylor polynomial. Then, using this expansion, we shall give the definition of τ-analyticity and formulate some properties of τ-analytic functionals.

The space Ξ. As in ([1]), $\xi(\tau)$ is a piecewise-continuous function on $D = [0;1]$ with values in $R = (-\infty, +\infty)$ (a trajectory), such that $\xi(\tau+0) = \xi(\tau)$, $\xi(1-0) = \xi(1)$. $\Xi = \{\xi(\tau)\}$. Elements of Ξ are denoted by ξ, possibly with an upper index. ξ_β^α will always denote $\xi^\alpha(t_\beta)$. $f(\xi)$ is a complex-valued functional on Ξ.

τ-Polynomial. In ([1]) it was established that the set of τ-polynomials coincides with the set of functionals representable in the form

$$f(\xi) = \int_0^1 \cdots \int_0^1 \varphi(t_1, \xi_1, \dots, t_n, \xi_n)\, dt_1 \dots dt_n = \int \varphi\, dt^n, \tag{1}$$

where $\varphi(t_1, x_1, \dots, t_n, x_n)$ is an arbitrary continuous complex-valued function of $2n$ real variables (the **density** of the functional), defined on $(D \times R)^n$; $\xi_j = \xi(t_j)$ (in accordance with the notation introduced above); t^n is Lebesgue measure on D^n. In what follows we shall everywhere assume that the density φ is a symmetric function of pairs of its arguments, i.e., for any substitution $p_n = (\alpha_1, \dots, \alpha_n)$ of the numbers $(1, \dots, n)$,

$$\varphi(t_{\alpha_1}, x_{\alpha_1}, \dots, t_{\alpha_n}, x_{\alpha_n}) = \varphi(t_1, x_1, \dots, t_n, x_n).$$

This does not restrict generality: if φ in (1) is not symmetric, then, denoting

$$S_n\varphi = \frac{1}{n!} \sum_{p_n} \varphi(t_{\alpha_1}, x_{\alpha_1}, \dots, t_{\alpha_n}, x_{\alpha_n}),$$

we obtain $f(\xi) = \int S_n\varphi\, dt^n$, where $f(\xi)$ is the same as in (1), while $S_n\varphi$ is symmetric.

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The totality of all τ-polynomials is a linear ring, which we shall denote by P.

The representation of $f(\xi) \in P$ in the form (1) is not unique. For example, the functional (1) will not change if to its density φ one adds a continuous symmetric function $\theta(t_1, \dots, t_n)$ (not depending on x_j) such that its mean value on D^n is equal to zero. Likewise, if one takes $\widetilde{\varphi} = S_{n+1}\varphi$, then the functional (1) is equal to $f(\xi) = \int \widetilde{\varphi}\, dt^{n+1}$.

Canonical expansion of a functional $f(\xi) \in P$. Consider an arbitrary function $\varphi(t_1, x_1, \dots, t_n, x_n)$, $t_j \in D$, $x_j \in R$. Fix $\xi^0 \in \Xi$. We shall call φ **simple relative to the trajectory** ξ^0 (more briefly, ξ^0**-simple**) if for any $j = 1, \dots, n$

$$\varphi(t_1, x_1, \dots, t_{j-1}, x_{j-1}, t_j, \xi_j^0, t_{j+1}, x_{j+1}, \dots, t_n, x_n) \equiv 0.$$

We shall call φ **degenerate** if it can be represented in the form $\varphi = \sum_{j=1}^n \varphi_j$, where the j-th summand does not depend on x_j.

Remark 1. If the density φ of the functional $f(\xi)$ in (1) is degenerate, then it can be represented in the form of an integral over D^{n-1}: $f(\xi) = \int \tilde{\varphi}\, dt^{n-1}$, where $\tilde{\varphi}$ is a continuous function on $(D \times R)^{n-1}$.

For brevity, introduce the notation

$$\varphi[x_{11} \pm x_{12} \pm \cdots] \cdots [x_{n1} \pm x_{n2} \pm \cdots] \tag{2}$$

($\pm$ denotes either plus or minus), which we shall understand as follows: first, regarding (2) as a product, expand the square brackets, preserving the order of the factors, and then, instead of each of the resulting terms of the form $\pm\varphi x_{1\beta_1} \cdots x_{n\beta_n}$, write $\pm\varphi(t_1, x_{1\beta_1}, \dots, t_n, x_{n\beta_n})$. For example:

$$\varphi[x_1] \cdots [x_n] = \varphi(t_1, x_1, \dots, t_n, x_n),$$

$$\varphi[x_1 - \xi_1][x_2] \cdots [x_n] = \varphi(t_1, x_1, t_2, x_2, \dots, t_n, x_n) - \varphi(t_1, \xi_1, t_2, x_2, \dots, t_n, x_n).$$

Denote

$$H\varphi = \varphi[x_1 - \xi_1^0] \cdots [x_n - \xi_n^0].$$

Lemma 1. For φ to be ξ^0-simple, it is necessary and sufficient that $\varphi = H\varphi$.

Lemma 2. If $H\varphi \equiv 0$ for some ξ^0, then φ is degenerate. From the degeneracy of φ it further follows that $H\varphi \equiv 0$ for any $\xi^0 \in \Xi$.

Corollary. If φ is simultaneously simple and degenerate, then $\varphi \equiv 0$.

Theorem 1. Every function $\varphi(t_1, x_1, \dots, t_n, x_n)$ can be represented uniquely in the form $\varphi = \varphi^1 + \varphi^2$, where φ^1 is ξ^0-simple and φ^2 is degenerate. Moreover,

$$\varphi^1 = H\varphi, \qquad \varphi^2 = \varphi - H\varphi.$$

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Example 1. Let $n = 2$, $\varphi = (x_1 + x_2)^3$, $\xi^0 \equiv 0$. Then

$$\varphi^1 = 3x_1x_2(x_1 + x_2), \qquad \varphi^2 = x_1^3 + x_2^3.$$

Example 2. If $n = 2$, $\varphi = \cos(x_1 + x_2)$, $\xi^0 \equiv 0$, then

$$\varphi^1 = \cos(x_1 + x_2) - \cos x_1 - \cos x_2 + 1, \qquad \varphi^2 = \cos x_1 + \cos x_2 - 1.$$

From Theorem 1 and Remark 1 it follows:

Theorem 2. Any τ-polynomial (1) can be represented in the form

$$f(\xi) = \int \psi^n(t_1, \xi_1, \dots, t_n, \xi_n)\, dt^n + \dots + \int \psi^1(t_1, \xi_1)\, dt^1 + \psi^0, \tag{3}$$

where $\psi^0 = f(\xi^0)$, and $\psi^1(t_1, x_1), \dots, \psi^n(t_1, x_1, \dots, t_n, x_n) = H\varphi$ are ξ^0-simple functions; moreover, for $j = 1, \dots, n-1$,

$$\psi^j = \binom{j}{n} \int_{D^{n-j}} \varphi[x_1 - \xi_1^0] \cdots [x_j - \xi_j^0][\xi_{j+1}^0] \cdots [\xi_n^0]\, dt_{j+1} \cdots dt_n. \tag{4}$$

Representation (3) will be called the canonical expansion of the τ-polynomial $f(\xi)$ about the trajectory ξ^0; the largest number s with non-identically-zero function ψ^s $(s \le n)$ is the degree of the τ-polynomial, and the functions $\psi^0, \dots, \psi^s$ are its canonical coefficients.

Simple τ-polynomial. If a τ-polynomial can be represented in the form

$$f(\xi) = \int \psi^s\, dt^s,$$

where ψ^s is a ξ^0-simple function, then it is called **simple relative to** ξ^0. Each summand on the right-hand side of (3) is a ξ^0-simple τ-polynomial. (The latter is true if $\xi^0(t)$ is continuous. In the case of discontinuous ξ^0, it may turn out that the function ψ^j in (3) does not satisfy the continuity condition imposed by us on the density of a τ-polynomial. However, this condition itself is not so essential.)

Theorem 3. If $f(\xi)$ is a ξ^0-simple τ-polynomial, then its τ-derivatives (see (1)) satisfy the equalities

$$f^{(j)}(\xi^0, t_1, x_1, \dots, t_j, x_j) \equiv 0 \qquad \text{for } j = 0, 1, \dots, s-1,$$

$$f^{(s)}(\xi^0, t_1, x_1, \dots, t_s, x_s) = s!\, \psi^s(t_1, x_1, \dots, t_s, x_s).$$

(Thus, a simple τ-polynomial is analogous to the function $(z - z^0)^s$, where z is a complex variable.)

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Theorem 4. The expansion (3) of an arbitrary τ-polynomial into a sum of ξ^0-simple τ-polynomials is unique (despite the nonuniqueness of the notation (1)).

Indeed, as follows from Theorem 3, the j-th canonical coefficient of a τ-polynomial is determined by the value of its j-th τ-derivative at $\xi = \xi^0$:

$$\psi^j = \frac{1}{j!} f^{(j)}(\xi^0, t_1, x_1, \dots, t_j, x_j). \tag{5}$$

If $\psi_1^1, \dots, \psi_1^m$ and $\psi_2^1, \dots, \psi_2^n$ $(m \le n)$ are the canonical coefficients, respectively, of the τ-polynomials f_1 and f_2, then the canonical coefficients of the sum $f_1 + f_2$ have the form

$$\psi^j = \psi_1^j + \psi_2^j \quad (j = 0, 1, \dots, n,\ \psi_1^j \equiv 0 \text{ for } j > m),$$

and the canonical coefficients of the product $f_1 \cdot f_2$ have the form

$$\psi^j = S_j \sum_{k=0}^{j} \psi_1^k(t_1, x_1, \dots, t_k, x_k) \cdot \psi_2^{j-k}(t_{k+1}, x_{k+1}, \dots, t_j, x_j),$$

where $j = 0, 1, \dots, m + n$.

Taylor' s formula for a τ-polynomial. Using equalities (5), the canonical expansion (3) of a τ-polynomial can be rewritten in the form

$$f(\xi) = f(\xi^0) + \sum_{j=1}^{s} \frac{1}{j!} \int f^{(j)}(\xi^0, t_1, \xi_1, \dots, t_j, \xi_j)\, dt^j. \tag{6}$$

We shall call equality (6) the **Taylor expansion** of the τ-polynomial $f(\xi)$ about the trajectory ξ^0.

Let us note that equalities (4) and (5) give an algebraic method for computing the τ-derivative of any order of a τ-polynomial, if its density φ is known.

Example 3. Let

$$f(\xi) = \int \cos(\xi_1 + \xi_2)\, dt^2, \qquad \xi^0 \equiv 0.$$

Then $f(0) = 1$,

$$f^{(1)}(0, t_1, x_1) = 2 \int_0^1 (\cos x_1 - 1)\, dt_2 = 2(\cos x_1 - 1),$$

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$$f^2(0, t_1, x_1, t_2, x_2) = \cos(x_1 + x_2) - \cos x_1 - \cos x_2 + 1.$$

τ-Analytic functionals. Let $f(\xi)$ be an arbitrary complex-valued functional on Ξ, infinitely τ-differentiable for every $\xi^0 \in \Xi$. If

$$f(\xi) = f(\xi^0) + \sum_{j=1}^{\infty} \frac{1}{j!} \int f^{(j)}(\xi^0, t_1, \xi_1, \dots, t_j, \xi_j)\, dt^j, \tag{7}$$

then $f(\xi)$ is called **τ-analytic** on Ξ, and the right-hand side of (7) is its Taylor series.

Example 4. Every τ-polynomial is τ-analytic.

Example 5. The functional

$$f(\xi) = \exp\left(\int_0^1 v(\xi(t))\, dt\right),$$

where $v(x)$ is a continuous function on R, is τ-analytic (but does not belong to P). Its j-th τ-derivative is equal to

$$f^{(j)} = \exp\left[\int_0^1 v(\xi^0(t))\, dt\right] \cdot \prod_{k=1}^{j} (v(x_k) - v(\xi_k^0)). \tag{8}$$

If we suppose that $v(0) \equiv 0$ and take $\xi^0 = 0$, then instead of (8) we obtain

$$f^j(0, t_1, x_1, \dots, t_j, x_j) = v(x_1) \cdots v(x_n).$$

The corresponding Taylor series has the form

$$f(\xi) = 1 + \sum_{j=1}^{\infty} \frac{1}{j!} \int v(\xi_1) \cdots v(\xi_j)\, dt^j.$$

The ring M of τ-analytic functionals. Take a sequence $\varphi^0, \varphi^1(t_1, x_1), \dots, \varphi^j(t_1, x_1, \dots, t_j, x_j), \dots$, where φ^j is a continuous symmetric function on $(D \times R)^j$, simple with respect to $\xi^0 = 0$, and satisfying the inequality $|\varphi^j| \le a^{j+1}$, where $0 < a < \infty$. Consider the series

$$\varphi^0 + \sum_{j=1}^{\infty} \frac{1}{j!} \int \varphi^j(t_1, \xi_1, \dots, t_j, \xi_j)\, dt^j = f(\xi),$$

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which converges for all ξ and defines a τ-analytic functional $f(\xi)$, with $f^{(j)}(0, t_1, x_1, \dots, t_j, x_j) = \varphi^j$.

Denote by $M(a)$ the totality of all such functionals, and put $M = \bigcup_a M(a)$. (The functional from Example 5 belongs to M, if $v(x)$ is bounded on R.)

Introduce in M a countable sequence of norms

$$\|f\|_n = \max_{0 \le j \le n} \sup_{\xi, t_k, x_k} \left| f^{(j)}(\xi, t_1, x_1, \dots, t_j, x_j) \right|,$$

where $n = 0, 1, 2, \dots$.

Take a sequence $f_1, \dots, f_m, \dots \in M$. We shall say that it **converges** to $f \in M$ **in the space** M, if there exists such an a that all $f_m \in M(a)$, and for every n

$$\lim_{m \to \infty} \|f - f_m\|_n = 0.$$

Then the limit of the sequence also belongs to $M(a)$.

Some properties of the space M.

1. M is a linear ring.
2. Every $f(\xi) \in M$ is bounded: if $f(\xi) \in M(a)$, then $|f(\xi)| \le ae^a$.
3. If $f(\xi) \in M(a) \subset M$, then $f(\xi + \xi^0) \in M(2ae^a) \subset M$ for any continuous $\xi^0 \in \Xi$.
4. The ring M is complete with respect to convergence in M.
5. Every $f(\xi) \in M$ is equal, in the sense of convergence in M, to the sum of its Taylor series.
6. $P_M = P \cap M$ is everywhere dense in M.
7. If $\{f_m\}$ converges to f in the space M, then for every ξ the equality

   $$\lim_{m \to \infty} f_m(\xi) = f(\xi)$$

   holds.

The convergence introduced in M turns this space into a union of countably normed spaces. It was used by the author for a generalization of the concept of measure. Elements of the space conjugate to M we call τ-measures. The concept of a τ-measure turns out to be convenient in considering such systems of finite-dimensional distributions in a functional space which cannot be extended to an ordinary measure in it (see (2)).

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CITED LITERATURE

1. E. V. Maikov, DAN, **155**, No. 2 (1964).

2. E. V. Maikov, UMN, **18**, issue 3, 243 (1963).

Note: Figure translations are in progress. See original paper for figures.

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