M. Kh. Zakhar-Itkin

1.** Consider the linear integral equation

1964

SovietRxiv

Abstract

Full Text

M. Kh. Zakhar-Itkin

On the Growth of the Eigenvalues of a Linear Integral Equation

(Presented by Academician P. S. Novikov on 26 VI 1964)

1. Consider the linear integral equation

$$y(x) = \lambda \int_0^1 K(x,t)y(t)\,dt. \tag{1}$$

The eigenvalues of this equation are the zeros of the entire function

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\int_0^1 \cdots \int_0^1 \Delta_n \begin{pmatrix} x_0, x_1, \dots, x_{n-1} \\ x_0, x_1, \dots, x_{n-1} \end{pmatrix} dx_0\, dx_1 \cdots dx_{n-1} \right) \lambda^n,$$

where

$$\Delta_n \begin{pmatrix} x_0, x_1, \dots, x_{n-1} \\ x_0, x_1, \dots, x_{n-1} \end{pmatrix} = \begin{vmatrix} K(x_0,x_0) & \cdots & K(x_0,x_{n-1}) \\ \cdots & \cdots & \cdots \\ K(x_{n-1},x_0) & \cdots & K(x_{n-1},x_{n-1}) \end{vmatrix}.$$

In [1] an estimate is given for the growth of $D(\lambda)$, under the assumption of analyticity of the kernel in the second variable on $[0,1]$, excluding the endpoints of the interval, where singularities of a special kind are possible.

We shall consider in the z-plane a domain with boundary Γ^*, symmetric with respect to the real axis and lying inside the circle $|z-\frac{1}{2}| \le \frac{1}{2}$. The boundary of the domain Γ^* touches the real axis at the points $(0,0)$ and $(1,0)$. Γ^* is the envelope of the family of oval curves $\Gamma_n = \Gamma[\alpha(n), q(n)]$, obtained under the mapping of the circle

$$w = \frac{1}{2} + \rho_n e^{i\theta}, \qquad 0 < \theta \le 2\pi,$$

$$\rho_n = \frac{1}{2} - \frac{[\alpha(n)]^{q(n)}}{[\alpha(n)]^{q(n)} + [1-\alpha(n)]^{q(n)}}, \qquad z(w) = \frac{w^{1/q(n)}}{w^{1/q(n)} + (1-w)^{1/q(n)}}.$$

The order of contact of Γ^* does not exceed p, if $q(n) = O(n^p)$ and $\alpha(n) = O(\frac{1}{\ln n})$.

Theorem 1. Suppose the following conditions are satisfied:

A. The kernel $K(x,z)$ is an analytic function of z inside the envelope Γ^* and on its boundary, with the exception of the points $(0,0)$, $(1,0)$ of the z-plane, $0 \le x \le 1$.

B. On the curve Γ_n the inequality

$$|K(x,z)| < \frac{e^{\gamma_1}[\alpha(n)]^{-k}}{[x(1-x)]^{1/2-\delta}}, \qquad 0 \le \delta < \frac{1}{2}, \quad k \text{ arbitrary}$$

holds.

C. In the square $0 \le x \le 1,\ 0 \le t \le 1$ the inequality

$$|K(x,t)| < \frac{e^{\gamma_0}}{[xt(1-x)(1-t)]^{1/2-\delta}}, \qquad \gamma_1 \ge \gamma_0 > 0.$$

holds.

If the order of contact of Γ^* does not exceed $1-2/i$, $i \ge 2$ arbitrary, then the eigenvalues of the integral equation (1) have the growth estimate

$$|\lambda_n| > e^{\theta \sqrt[i]{n} + \gamma},$$

where θ, γ are certain constants.

If $i=2$, then $K(x,z)$ is an analytic function of z in some circular lune containing the interval $0 < \operatorname{Re} z < 1$; Theorem 1 reduces to Theorem XI of A. O. Gel′fond ([1]).

In the proof of Theorem 1 the following estimate is basic (Theorem X of ([1])):

$$\int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{s=0}^{n} (1-x_s^2)^{1/q(n+1)-1} \begin{vmatrix} \dfrac{1}{1-\mu^2 x_0^2} & \cdots & \dfrac{1}{1-\mu^2 x_0 x_n} \\ \cdots & \cdots & \cdots \\ \dfrac{1}{1-\mu^2 x_0 x_n} & \cdots & \dfrac{1}{1-\mu^2 x_n^2} \end{vmatrix} dx_0 \cdots dx_n <$$

$$< \frac{C^2}{(1-\mu)^{n+1}} \exp\left[-\frac{\pi\sqrt{2}}{10\sqrt{q(n+1)}} (n+1)^{3/2} \right].$$

Here $\mu < 1,\ \varepsilon_0 \ge 1/q(n+1) > 0$. Its application makes it possible to obtain the estimate of the growth of $D(\lambda)$

$$\max_{|\lambda|=r} |D(\lambda)| < \sum_{n=0}^{\infty} r^n \exp\left[-\theta n^{3/2-p/2} + O(n \ln n)\right],$$

whence, with the aid of Jensen's inequality, we obtain an estimate for the growth of the zeros of this entire function.

In connection with Theorem 1 it is interesting to mention ([2],[3]) that if the kernel $K(x,z)$ is analytic in z on the whole interval $[0,1]$, including its endpoints, uniformly in x, then $|\lambda_n| > e^{\theta n+\gamma}$.

2. Let us consider (1) under other assumptions concerning the kernel $K(x,t)$. We shall assume that $K(x,t)$ is a function of Green' s-function type:

1) The function $K(x,t)$, its partial derivatives with respect to x up to order $n+2$, and its partial derivative $[K_x^{(n)}(x,t)]_t'$ exist, are continuous and bounded in the square $0 \le x \le 1,\ 0 \le t \le 1$, with the exception of the diagonal of the square $x=t$, on which

$$K_x^{(i)}(x+0,t) - K_x^{(i)}(x-0,t) = r_i(x).$$

Consider the simplest case $r_{n+1}(x) \equiv 1,\ r_i(x) \equiv 0,\ i \le n+1$.

We make the additional assumptions:

2) The integral equation

$$y(x) + \int_0^1 R(x,t) y(t)\, dt = 0$$

with kernel

$$R(x,t) = K_x^{(n)}(x,t) - \delta(x-t)$$

has only the trivial solution.

3) For $x=0$ the equalities hold $(i = 0, 1, \dots, n-1)$

$$K_x^{(i)}(x,1) - \int_0^1 K_x^{(i)}(x,t)(E+L)^{-1} R(t,1)\, dt =$$

$$= K_x^{(i)}(x,0) - \int_0^1 K_x^{(i)}(x,t)(E+L)^{-1} R(t,0)\, dt + \gamma_i.$$

Here

$$\gamma_i = \begin{cases} 0, & 0 \le i \le n-2, \\ 1, & i = n-1; \end{cases} \qquad Ly(x) = \int_0^1 R(x,t)y(t)\,dt.$$

Theorem 2. *If conditions 1), 2), 3) are fulfilled, the integral equation (1) has a sequence (one or several, but not more than n sequences) of eigenvalues, whose asymptotic values are the quantities $(2\pi i m)^n + O(m^{n-2})$, $m > m_0$.*

For $n = 1, 2$ Theorem 2 was proved by G. M. Mordasova ([4]).

We seek solutions of equation (1) among the solutions of the integro-differential equation

$$y^{(n)}(x) = \lambda y(x) + \lambda \int_0^1 R(x,t)y(t)\,dt, \tag{2}$$

obtained from (1) by n-fold differentiation. A solution $y(x)$ of equation (2) will be an eigenfunction of equation (1) if and only if the n conditions

$$y^{(i)}(0) = \lambda \int_0^1 K_x^{(i)}(0,t)y(t)\,dt, \qquad i = 0, 1, \dots, n-1. \tag{3}$$

are satisfied.

We shall find the general solution of equation (2) in the form

$$y(x) = \sum_{i=1}^{n} C_i e^{\mu_i x} + \int_0^1 \chi(x,t)y(t)\,dt,$$

where $\mu_1, \mu_2, \dots, \mu_n$ are the roots of degree n of λ, and

$$\chi(x,t) = \sum_{i=1}^{n} \left\{ -\frac{1}{n} \left[R(x,t) + \frac{1}{\mu_i} R'_x(x,t) + \begin{cases} \dfrac{e^{\mu_i(x-t)}}{\mu_i n}, & x < t, \\ 0, & x \ge t \end{cases} + O\left(\frac{1}{\mu_i^2}\right) \right] \right\}$$

is a particular solution of the equation

$$\chi_x^{(n)}(x,t) = \lambda \chi(x,t) + \lambda R(x,t),$$

obtained from the general solution

$$\chi(x,t) = \sum_{i=1}^{n} \left[d_i e^{\mu_i x} + \frac{\mu_i}{n} \int_0^x R(s,t) e^{\mu_i(x-s)}\,ds \right]$$

for a special choice of $d_i,\ i = 1, 2, \dots, n$. Since

$$\chi'_x(x,t) = \sum_{i=1}^{n} \mu_i \left[d_i e^{\mu_i x} + \frac{\mu_i}{n} \int_0^x R(s,t) e^{\mu_i (x-s)}\, ds \right],$$

we have

$$y'(x) = \sum_{i=1}^{n} \mu_i \left[C_i e^{\mu_i x} + \int_0^1 \left(d_i e^{\mu_i x} + \frac{\mu_i}{n} \int_0^x R(s,t) e^{\mu_i (x-s)}\, ds \right) y(t)\, dt \right].$$

Using condition 2) and the general theorems of the theory of linear operators, we obtain the asymptotic representation for the solution of equation (2)

$$\begin{aligned} y(x) &= \sum_{i=1}^{n} \left[C_i e^{\mu_i x} + \int_0^1 \left(d_i e^{\mu_i x} + \frac{\mu_i}{n} \int_0^x R(s,t) e^{\mu_i (x-s)}\, ds \right) y(t)\, dt \right] \\ &= \sum_{i=1}^{n} C_i \left\{ e^{\mu_i x} - (E+L)^{-1} \frac{1}{\mu_i} \left[(R(x,1) e^{\mu_i} - R(x,0)) + \frac{n-1}{n} \sigma(x,t) e^{\mu_i x} \right.\right. \\ &\qquad \left.\left. - \int_0^1 \sigma(x,t) e^{\mu_i t} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\mu_i e^{\mu_j (x-t)}}{\mu_i - \mu_j}\, dt \right] + O\left(\frac{1}{\mu_i^2} \right) \right\} \\ &= \sum_{i=1}^{n} C_i \left\{ e^{\mu_i x} + A\left(\frac{1}{\mu_i}, x \right) \right\}. \end{aligned}$$

Here

$$\sigma(x,t) = \begin{cases} -1, & x < t, \\ 0, & x \geq t, \end{cases} \quad \text{for } \operatorname{Re} \mu_i \geq 0, \qquad \sigma(x,t) = \begin{cases} 0, & x < t, \\ 1, & x \geq t, \end{cases} \quad \text{for } \operatorname{Re} \mu_i < 0.$$

Similarly,

$$y^{(k)}(x) = \sum_{i=1}^{n} C_i \mu_i^{k-1} \left\{ e^{\mu_i x} + A\left(\frac{1}{\mu_i}, x \right) \right\}, \qquad k = 1, 2, \dots, n-1.$$

The conditions (3) that the solution of equation (2) be an eigenfunction of equation (1) can be rewritten in the form

$$\sum_{i=1}^{n} C_i \mu_i^{n-1} \left\{ \left[K(0,1) - \int_0^1 K(0,t)(E+L)^{-1} R(t,1)\,dt \right] e^{\mu_i} - \left[K(0,0) - \int_0^1 K(0,t)(E+L)^{-1} R(t,0)\,dt \right] + O \right.$$

$$\sum_{i=1}^{n} C_i \mu_i^{n-2} \left\{ \left[K_x'(0,1) - \int_0^1 K_x'(0,t)(E+L)^{-1} R(t,1)\,dt \right] e^{\mu_i} - \left[K_x'(0,0) - \int_0^1 K_x'(0,t)(E+L)^{-1} R(t,0)\,dt \right] \right.$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\sum_{i=1}^{n} C_i \mu_i \left\{ \left[K_x^{(n-2)}(0,1) - \int_0^1 K_x^{(n-2)}(0,t)(E+L)^{-1} R(t,1)\,dt \right] e^{\mu_i} - \left[K_x^{(n-2)}(0,0) - \int_0^1 K_x^{(n-2)}(0,t)(E+L)^{-1} \right.\right.$$

$$\sum_{i=1}^{n} C_i \left\{ \left[K_x^{(n-1)}(0,1) - \int_0^1 K_x^{(n-1)}(0,t)(E+L)^{-1} R(t,1)\,dt \right] e^{\mu_i} - \left[1 + K_x^{(n-1)}(0,0) - \int_0^1 K_x^{(n-1)}(0,t)(E+L) \right.\right.$$

Conditions 3) allow us to rewrite the necessary and sufficient condition for the existence of a solution of equation (1) in the form

$$\begin{vmatrix} \mu_1^{n-1}\left[e^{\mu_1} - 1 + O(1/\mu_1)\right] & \dots & \mu_n^{n-1}\left[e^{\mu_n} - 1 + O(1/\mu_n)\right] \\ \mu_1^{n-2}\left[e^{\mu_1} - 1 + O(1/\mu_1)\right] & \dots & \mu_n^{n-2}\left[e^{\mu_n} - 1 + O(1/\mu_n)\right] \\ \dots\dots\dots\dots\dots\dots\dots & & \\ \mu_1\left[e^{\mu_1} - 1 + O(1/\mu_1)\right] & \dots & \mu_n\left[e^{\mu_n} - 1 + O(1/\mu_n)\right] \\ e^{\mu_1} - 1 + O(1/\mu_1) & \dots & e^{\mu_n} - 1 + O(1/\mu_n) \end{vmatrix} = 0.$$

If for no μ_k, $k = 1, 2, \dots, n$, is the equality $\mu_k \sim 2\pi i m$ satisfied, then on the left we obtain asymptotically a Vandermonde determinant, not equal to zero, since all μ_k are distinct. If $\mu_k \sim 2\pi i m + O\left(\frac{1}{m}\right)$, then the condition that the determinant be equal to zero is satisfied.

Theorems 1 and 2 are close in meaning. The eigenvalues of equation (1) grow the faster, the smoother the kernel $K(x,t)$ is, and greater smoothness may be understood in the sense of smaller growth when tending to the singular point and in the sense of continuous differentiability up to a higher order.

The author expresses gratitude to A. O. Gelfond for supervising the work.

Moscow State University
named after M. V. Lomonosov

Received
26 VI 1964

REFERENCES

1. A. O. Gelfond, Appendix to the book by V. Lovitt, *Linear Integral Equations*, Moscow, 1957.

2. A. O. Gelfond, *C. R.*, **192**, 828 (1931).

3. E. Hille, J. Tamarkin, *Acta Math.*, **57**, 1 (1931).

4. G. M. Mordasova, *DAN*, **142**, No. 5, 1022 (1962).

Note: Figure translations are in progress. See original paper for figures.