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**Abstract**

**Full Text**

**Mathematics**

**A. M. Vinogradov**

## **Some homotopic properties of the space of embeddings of a circle in a sphere or ball**

*(Presented by Academician P. S. Aleksandrov, January 20, 1964)*

Let  $M^m, N^n$  be smooth manifolds. Denote by  $B\lambda(M^m, N^n)$  the space of all smooth embeddings of the first manifold into the second, endowed with the  $C^1$ -topology. The problem of classifying embeddings  $M^m \rightarrow N^n$  consists in describing the set  $\pi_0(B\lambda(M^m, N^n))$ . In the present work we attempt to go further, i.e., to compute the groups  $\pi_i(B\lambda(M^m, N^n))$ ,  $i > 0$ , for the case when  $M^m$  is the one-dimensional sphere  $S^1$ , and  $N^n$  is either  $S^n$  (the  $n$ -sphere), or  $D^n$  (the  $n$ -ball) (or, what is the same, the Euclidean space  $R^n$ ).

**Theorem 1.** *For  $i \leq 2n - 7$  there is an isomorphism*

$$\pi_i(B\lambda(S^1, S^n)) = \pi_i(T^1(S^n)),$$

where  $T^1(S^n)$  is the space of nonzero tangent vectors of the sphere  $S^n$ .

For example, the space  $B\lambda(S^1, S^4)$  is simply connected (although, since the space  $B\lambda(S^1, S^3)$  is disconnected, it would be more natural to suppose that it is only connected).

**Theorem 2.** *The group  $\pi_i(B\lambda(S^1, R^n))$  contains as a direct summand the group  $\pi_i(T^1(S^{n-1}))$ . If  $i \leq 2n - 7$ , then*

$$\pi_i(B\lambda(S^1, R^n)) = \pi_i(T^1(S^{n-1})).$$

Comparison of Theorems 1 and 2 indicates a great difference between the spaces  $B\lambda(S^1, R^n)$  and  $B\lambda(S^1, S^n)$ . On the other hand, the natural embedding

$$B\lambda(S^1, S^{n-1}) \rightarrow B\lambda(S^1, R^n),$$

generated by the embedding  $S^{n-1} \rightarrow R^n$ , induces a monomorphism of homotopy groups which, in dimensions  $\leq 2n - 7$ , is also an epimorphism.

In connection with Theorem 1 the question naturally arises: what are the groups  $\pi_i(B\lambda(S^1, S^n))$  for  $i \geq 2n - 6$ ? One may think that these groups, in particular  $\pi_{2n-6}(B\lambda(S^1, S^n))$ , are not even of finite type (cf.  $\pi_0(B\lambda(S^1, S^3))$ ).

Below a sketch will be given of the proofs of Theorems 1 and 2. Before this, however, I wish to note that the methods used here admit a generalization which

makes it possible, to some extent, to study the spaces  $B\lambda(S^k, S^n)$ ,  $k \geq 1$ , and also some other spaces of smooth and combinatorial embeddings.

**Proof of Theorem 1.** Fix on the circle  $S^1$  a point  $x_0$ , and to every embedding  $f \in B\lambda(S^1, S^n)$  assign the differential  $df|_{x_0}$ . As a result we obtain a mapping

$$\Phi : B\lambda(S^1, S^n) \xrightarrow{B\lambda_0(S^1, S^n)} T^1(S^n),$$

where the set  $B\lambda_0(S^1, S^n)$  consists of all embeddings  $f \in B\lambda(S^1, S^n)$  with fixed differential at the point  $x_0$ . Below it will be shown that  $\pi_i(B\lambda_0(S^1, S^n)) = 0$  if  $i \leq 2n - 7$ . Since the mapping  $\Phi$  is a Serre fibration, Theorem 1 follows from this result.

Let  $I = [0, 1]$  and  $\alpha = (1/2, \dots, 1/2, 0) \in \partial I^n$ ,  $\beta = (1/2, \dots, 1/2, 1) \in \partial I^n$ . Denote by  $Q_{k,n}$  the space of all simple  $k$ -segment polygonal lines, lying

lying in the cube  $I^n$  and joining the points  $\alpha$  and  $\beta$ . Dividing the first segment of a  $k$ -segment polygonal line in half, we obtain a mapping  $\varphi_k : Q_{k,n} \rightarrow Q_{k+1,n}$ . Denote by  $Q_n$  the limit of the sequence

$$Q_{1,n} \xrightarrow{\varphi_1} Q_{2,n} \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{k-1}} Q_{k,n} \xrightarrow{\varphi_k} \dots$$

Then the following holds:

**Lemma 1.** The spaces  $B\lambda_0(S^1, S^n)$  and  $Q_n$  are homotopy equivalent.

Therefore it is enough to show that  $\pi_i(Q_{k,n}) = 0$ , if  $i \leq 2n - 7$ .

Let  $P_{j,n}$  be the space of simple polygonal lines beginning at the point  $\alpha$ . Then, assigning to each polygonal line  $g \in Q_{k,n}$  the polygonal line consisting of its first  $j$  segments, we obtain a mapping  $p_j : Q_{k,n} \rightarrow P_{j,n}$ . Put  $Q_{k,n}^j = p_j(Q_{k,n})$ . Then the equality  $p_{j,q}j = p_{j-1}$  uniquely defines a mapping  $q_j : Q_{k,n}^j \rightarrow Q_{k,n}^{j-1}$ .

It is now not hard to show that the following is true.

**Lemma 2.** The mapping  $q_k$  is a homeomorphism; all fibers of the mapping  $q_{k-1}$  are  $(n - 4)$ -connected; all fibers of the mapping  $q_j$ ,  $j < k - 1$ , are homotopically trivial.

On the basis of what has been said one can establish that the space  $Q_{k,n}^{k-2}$  is homotopically trivial, and the space  $Q_{k,n}^{k-1} \cong Q_{k,n}$  is  $(n - 4)$ -connected.

Consider now the mapping  $q_{k-1}$  and its spectral sequence  $\{E_{p,q}^r\}$  of singular homologies. Then  $E_{p,q}^2 = H_p(Q_{k,n}^{k-2}; \mathcal{H}_q)$ , where  $\mathcal{H}_q$  is a certain system of coefficients. From Lemma 2 it follows that  $\mathcal{H}_q = 0$ , if  $q \neq 0$ ,  $q \neq n - 2$ ,  $q \neq n - 3$ , and since the space  $Q_{k,n}^{k-2}$  is homotopically trivial,  $E_{0,q}^2 = 0$ , if  $q > 0$ ,  $n > 4$ .

The coefficient systems  $\mathcal{H}_{n-2}$ ,  $\mathcal{H}_{n-3}$  are already nontrivial. However, one can show that  $E_{n-2,q}^2 = 0$ , if  $q \leq n - 5$ ;  $E_{n-3,q}^2 = 0$ , if  $q \leq n - 4$ . For example, the triviality of the group  $E_{n-2,q}^2$ ,  $q \leq n - 5$ , follows from Lemma 3.

**Lemma 3.** Let  $X = \{x \in Q_{k,n}^{k-2} \mid q_{k-1}^{-1}(x) \text{ is homotopically trivial}\}$ , and let  $L$  be a finite polyhedron of dimension  $\leq n - 5$ . Then for an arbitrary mapping  $f : L \rightarrow Q_{k,n}^{k-2}$  there exists a mapping  $f' : L \rightarrow Q_{k,n}^{k-2}$ , arbitrarily close to  $f$ , such that  $f'(L) \subset X$ .

Finally, since  $E_{p,q}^2 = E_{p,q}^\infty$ ,  $n \geq 4$ , it follows from the triviality of the groups  $E_{p,q}^2$ ,  $0 < p + q < 2n - 7$ , Lemma 2 and Hurewicz' s theorem that Theorem 1 follows.

**Remark.** One can show that, generally speaking,  $H_{2n-6}(Q_{k,n}) \neq 0$ .

**Proof of Theorem 2.** Theorem 2 is a consequence of the more general Theorem 3.

Put  $\tilde{N}^n = N^n \setminus x_0$ ,  $x_0 \in \text{int } N^n$ . Then the following holds.

**Theorem 3.** Let  $N^n = S^n$ . Then

$$\pi_i(B\Lambda_0(S^k, \tilde{N}^n)) = \pi_i(B\Lambda(S^k, N^n)) + \pi_i(S^{n-k-1}).$$

We shall first show that Theorem 2 follows from Theorem 3. To this end consider the fibrations

$$\Phi : B\Lambda(S^k, R^n) \xrightarrow{B\Lambda(S^k, R^n)} R^n \times V_{n,k}, \quad p : V_{n,k+1} \xrightarrow{S^{n-k-1}} V_{n,k}$$

and note that every frame  $\xi \in V_{n,k-1}$  defines in a natural way an embedding  $q(\xi) \in B\Lambda(S^k, R^n)$ , which realizes the sphere  $S^k$  in the "standard" way in the  $(k + 1)$ -plane spanned by the frame  $\xi$ .

The mapping  $q$  induces the following commutative diagram

$$\begin{array}{ccccccc} \cdots \rightarrow (B\Lambda_0(S^k, R^n)) & \rightarrow & \pi_i(B\Lambda(S^k, R^n)) & \xrightarrow{\Phi_*} & \pi_i(R^n \times V_{n,k}) & \rightarrow & \cdots \\ & \uparrow s_* & \uparrow q_* & & \uparrow t_* & & \\ \cdots \rightarrow \pi_i(S^{n-k-1}) & \rightarrow & \pi_i(V_{n,k+1}) & \xrightarrow{p_*} & \pi_i(V_{n,k}) & \rightarrow & \cdots \end{array}$$

in which  $t_*$  is an isomorphism,  $s_*$  is a monomorphism, defined by the decomposition

$$\pi_i(B\Lambda_0(S^k, R^n)) = \pi_i(B\Lambda_0(S^k, S^n)) + \pi_i(S^{n-k-1}).$$

It follows easily from this that  $q_*$  is a monomorphism. Taking  $k = 1$  and applying Theorem 1, we obtain Theorem 2.

**Proof of Theorem 3.** Let one of the components  $\tilde{S}^{n-1}$  of the boundary  $\partial L^n$  of the manifold  $L^n$  be diffeomorphic to the sphere  $S^{n-1}$ , and let  $\tilde{S}^{k-1}$  be the image

in  $\tilde{S}^{n-1}$ , under this diffeomorphism, of the sphere  $S^{k-1}$  standardly embedded in  $S^{n-1}$ . Consider the space  $X_k(\tilde{S}^{n-1}, L^n)$  of all embeddings of the ball  $D^k$  in  $L^n$  satisfying the conditions:

- a) the embedding  $f|_{\partial D^k} = g$  does not depend on  $f \in X_k(\tilde{S}^{n-1}, L^n)$ ,  $g(\partial D^k) = \tilde{S}^{k-1}$ ;
- $\beta$ ) the differential  $df|_{\partial D^k}$  does not depend on  $f \in X_k(\tilde{S}^{n-1}, L^n)$  and is transversal to  $\partial L^n$ .

Let  $N^n$  be some manifold,  $x \in \text{int } N^n$ . Cutting out from  $N^n$  the interior of a small ball  $D_x^n$  with center at  $x$ , we obtain the manifold  $L^n = L(N^n)$ . As the component  $\tilde{S}^{n-1}$  of the boundary  $\partial L^n$  of this manifold we choose  $\partial D_x^n$ , and put  $X_k(N^n) = X_k(\tilde{S}^{n-1}, L(N^n))$ . The following lemma holds:

**Lemma 4.** *The spaces  $B\Lambda_0(S^k, N^n)$  and  $X_k(N^n)$  are homotopy equivalent.*

We shall call an embedding

$$\varphi : D^k \times D^{n-k} \rightarrow L(N^n)$$

**admissible** if  $\varphi|_{D^k \times 0} \in X_k(N^n)$  and  $\varphi(\partial D^k \times D^{n-k}) \subset \tilde{S}^{n-1}$ . Two admissible embeddings  $\varphi, \varphi'$  will be called **equivalent** if there exists a mapping  $\psi : D^k \rightarrow SO(n-k)$  such that  $\varphi(x, y) = \varphi'(x, \psi(x) \cdot y)$ ,  $x \in D^k$ ,  $y \in D^{n-k}$ . The class of admissible equivalent embeddings  $\varphi$  for which  $\varphi|_{D^k \times 0} = f$  will be called a **tubular neighborhood** of the embedding  $f \in X_k(N^n)$ .

Let  $B \subset X_k(N^n)$  be compact and  $N^n = \tilde{N}^n \cup x_0$ .

**Lemma 5.** *There exists a family of tubular neighborhoods  $u_f$ , continuous relative to  $f$ ,  $f \in B$ , containing within themselves the point  $x_0$ , if  $N^n = S^n$ .*

Every embedding

$$s : D^k \rightarrow L(N^n)$$

having the form  $\varphi \cdot s'$ , where  $s'$  is a sectioning surface of the fibration

$$D^k \times D^{n-k} \rightarrow D^k,$$

and the mapping

$$\varphi : D^k \times D^{n-k} \rightarrow L(N^n)$$

is admissible, will be called a **sectioning surface** of the tubular neighborhood determined by the embedding  $\varphi$ . We may regard the space  $L(\tilde{N}^n)$  as a subspace of the space  $L(N^n)$ . Under this condition the following is true:

**Lemma 6.** *Let  $B \subset L(\tilde{N}^n)$  be some compact set, and let the family  $u_f$ ,  $f \in B$ , be the same as in Lemma 5. Then there exists a continuous family  $s_f$  of nonzero sectioning surfaces of the family  $u_f$ , passing through the point  $x_0$ .*

Let now  $N^n = S^n$ ,  $\mu : S^i \rightarrow X_k(N^n)$  be some mapping,  $B = \mu(S^i)$ , and  $a \in \partial D^k$ . Then, putting

$$H(y) = s_{\mu(y)}(a) \in \tilde{S}^{n-1} \setminus \tilde{S}^{k-1}, \quad y \in S^i,$$

we obtain a mapping

$$H : S^i \rightarrow \tilde{S}^{n-1} \setminus \tilde{S}^{k-1}.$$

**Lemma 7.** *The homotopy class of the mapping  $H$  depends only on the homotopy class of the mapping  $\mu$  and determines an epimorphism*

$$h : \pi_i(X_k(\tilde{N}^n)) \rightarrow \pi_i(\tilde{S}^{n-1} \setminus \tilde{S}^{k-1}) \approx \pi_i(S^{n-k-1}).$$

Let, further, the embedding  $\chi : I \rightarrow L(N^n)$  be such that  $\chi(0) = x_0$ ,  $\chi(1) \in \tilde{S}^{n-1} \setminus \tilde{S}^{k-1}$ , and  $\chi((0, 1)) \cap \partial L^k = \emptyset$ . Let  $\bar{X}_k(N^n) \subset X_k(S^n)$  consist of those embeddings  $f \in X_k(N^n)$  for which  $f(D^k) \cap \chi(I) = \emptyset$ . Then the following holds.

**Lemma 8.** *The space  $X_k(N^n)$  contracts onto  $\bar{X}_k(N^n) \subset X_k(N^n)$ .*

Since  $\bar{X}_k(N^n) \subset X_k(\tilde{N}^n)$ , it follows from Lemma 8 that  $\pi_i(X_k(N^n))$  is a direct summand of the group  $\pi_i(X_k(\tilde{N}^n))$ . Moreover, it is not hard to show that  $\pi_i(\bar{X}_k(N^n)) = \text{Ker } h$ . Therefore, by Lemma 7,

$$\pi_i(X_k(\tilde{N}^n)) = \pi_i(\bar{X}_k(N^n)) + \pi_i(S^{n-k-1}),$$

which proves Theorem 3.

Moscow Institute  
of Radio Electronics and Mining Electromechanics

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*Note: Figure translations are in progress. See original paper for figures.*

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