Reports of the Academy of Sciences of the USSR

Corresponding Member of the Academy of Sciences of the USSR I. M. GELFAND, M. I. GRAEV,

1964

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1964. Volume 156, No. 3

MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR I. M. GELFAND, M. I. GRAEV,
I. I. PIATETSKII-SHAPIRO

REPRESENTATIONS OF ADELE GROUPS

1. Let G be a linear algebraic group of matrices over a field K of algebraic numbers of finite degree. Let us recall the definition of the **adele group** of the group G (see ([1])). Consider all inequivalent norms $|x|_{\mathfrak{p}}$ in the field K. Let $G_{\mathfrak{p}}$ be the completion of the group G with respect to the norm $|x|_{\mathfrak{p}}$; let $U_{\mathfrak{p}}$ be the subgroup of integral matrices in $G_{\mathfrak{p}}$ (i.e., those matrices $\|x_{ij}\|$ in $G_{\mathfrak{p}}$ for which $|x_{ij}|_{\mathfrak{p}} \leqslant 1$ for all i, j). Consider the Tychonoff product

$$\hat{G} = \prod_{\mathfrak{p}} G_{\mathfrak{p}}$$

of the groups $G_{\mathfrak{p}}$. The group of adeles is the subgroup

$$G_A \subset \hat{G}$$

of elements

$$\prod_{\mathfrak{p}} g_{\mathfrak{p}},$$

where $g_{\mathfrak{p}} \in U_{\mathfrak{p}}$ for all $\mathfrak{p}$, except possibly for finitely many. We note that the set of elements $\prod g_{\mathfrak{p}}$ of G_A, where all $g_{\mathfrak{p}}$ are equal to one another and belong to G, forms in G_A a discrete subgroup Γ (the subgroup of **principal adeles**), isomorphic to G. In what follows we shall identify the group G with the group of principal adeles $\Gamma \subset G_A$.

An important role in many questions is played by the homogeneous space G_A/Γ ([1]). In the present paper representations of the group of adeles G_A are studied. The main result is a description of the spectrum of the representation of the group $G_A{}^*$ in the space of functions on G_A/Γ, where G is the group of unimodular matrices of order 2 over the field of rational numbers (Theorem 3). In particular, a connection is established between the structure of this spectrum and the properties of Dirichlet L-functions. Namely, it turns out that the known functional relation for L-functions is equivalent to the assertion that the continuous spectrum is simple.

2. Impose on the group G the following condition. **For all $\mathfrak{p}$ (except possibly for finitely many of them), in the space of any irreducible unitary representation of the group $G_{\mathfrak{p}}$ there is at most one vector invariant with respect to the subgroup $U_{\mathfrak{p}}$.** It can be shown that this condition is certainly satisfied for a sufficiently broad class of groups** (in particular, for all Dickson–Chevalley groups). An irreducible unitary representation of the group $G_{\mathfrak{p}}$ is called a **representation of class 1** if in the representation space there exists a vector invariant with respect to the subgroup $U_{\mathfrak{p}}$.

We shall describe the construction of irreducible unitary representations of the group G_A. Suppose that for each $\mathfrak{p}$ we are given an irreducible unitary representation $T_{\mathfrak{p}}(g_{\mathfrak{p}})$ of the group $G_{\mathfrak{p}}$, and that all $T_{\mathfrak{p}}$, except for finitely many of them, are representations of class 1. Denote by $H_{\mathfrak{p}}$ the space in which $T_{\mathfrak{p}}(g_{\mathfrak{p}})$ acts. Choose in $H_{\mathfrak{p}}$ an orthonormal basis $\xi_{\mathfrak{p}k}$, $k = 1, 2, \ldots$; in the case of a representation of class 1, as $\xi_{\mathfrak{p}1}$ we agree to take a vector invariant with respect to the operators $T_{\mathfrak{p}}(u_{\mathfrak{p}})$, $u_{\mathfrak{p}} \in U_{\mathfrak{p}}$. Construct a new Hilbert space H, in which the orthonormal–

* By the spectrum of a representation here is meant the set of irreducible representations entering into the decomposition of the given representation.

** Since, as in the case of Lie groups ([3]), one can prove that, for this condition to hold, it is sufficient that there exist an involutive antiautomorphism of the group $G_{\mathfrak{p}}$ carrying the subgroup $U_{\mathfrak{p}}$ into itself.

a basis are all possible formal products $\xi = \prod_p \xi_{pi_p}$, where $i_p = 1$ for all p, except for a finite number. We define the representation operator $T(g)$, $g = \prod g_p$, in the space H by the formula $T(g)\xi = \prod T_p(g_p)\xi_{pi_p}$.

We shall call the constructed representation $T(g)$ the **tensor product** of the representations $T_p(g_p)$ and write $T(g) = \prod \otimes T_p(g_p)$.

Theorem 1. *The representation $T(g)$ is irreducible. Conversely, every irreducible unitary representation of the group G_A can be obtained by the construction described.*

Remark. An analogous assertion is valid for the product, in the sense of Vilenkin, of arbitrary locally compact groups with distinguished subgroups in them.

3. Consider the space H of functions $f(g)$ on G_A satisfying the following conditions: $f(\gamma g) = f(g)$, for any $\gamma \in \Gamma$; $\|f\|^2 = \int_F |f(g)|^2 \, dg < +\infty$, where F is a fundamental domain of the subgroup Γ in G_A. We construct in H a representation $T_0(g)$ of the group G_A, defining the representation operators as shift operators: $T_0(g_0)f(g) = f(gg_0)$. Below two results on the structure of the spectrum of the representation $T_0(g)$ are formulated.

A. *Consider the restriction* $T_0(g_{p_1,\dots,p_k})$ *of the representation* $T_0(g)$ *to the subgroup* $G_{p_1,\dots,p_k} = G_{p_1} \times \dots \times G_{p_k}$. *The irreducible unitary representations of the group* $G_{p_1,\dots,p_k}$ *entering into the decomposition of the representation* $T_0(g_{p_1,\dots,p_k})$ *into irreducibles form an everywhere dense set in the totality of all irreducible unitary representations of the group* $G_{p_1,\dots,p_k}$.

B. *If in an irreducible unitary representation* $\prod \otimes T_p(g_p)$ *of the group* G_A *at least one factor* $T_p(g_p)$ *is the identity representation and* G_p *is not compact, then this representation is not contained in the discrete spectrum of the representation* $T_0(g)$. *In other words, it cannot be realized in any closed subspace of the representation space* $T_0(g)$.

The proof of these results is based on the following remark. If G_p is not compact, then the set ΓG_p is everywhere dense in the group G_A.

4. The further results concern the case when G is the group of unimodular matrices of the second order over the field of rational numbers. In this case the adele group G_A consists of elements of the form $g = (g_\infty, g_2, \dots, g_p, \dots)$, where g_∞ is a real unimodular matrix, g_p is a unimodular matrix over the field of p-adic numbers, and the elements of the matrices g_p, beginning with some p, are integral p-adic numbers.

Let Z be the subgroup of matrices of the form $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$, D the subgroup of diagonal matrices; Z_A, D_A the corresponding adele groups. Consider the homogeneous space $X = G_A / Z_A D$ of right cosets $Z_A D g$ of the group G_A with respect to the subgroup $Z_A D$. It is not difficult to see that X is isomorphic to the set of sequences of the form $x = (x_\infty, x_2, \dots, x_p, \dots)$, where $x_p = (x_p^1, x_p^2)$ is a two-dimensional vector whose components are p-adic numbers, with $|x_p| = 1$ for sufficiently large p; moreover the sequences $(x_\infty, x_2, \dots, x_p, \dots)$ and $(\lambda x_\infty, \lambda x_2, \dots, \lambda x_p, \dots)$, where λ is any rational number, are identified.

Denote by xg the result of applying the transformation $g \in G_A$ to an element $x \in X$. Let further Λ be the group of rational numbers under multiplication, Λ_A its adele group. For any $\lambda = (\lambda_\infty, \lambda_2, \dots, \lambda_p, \dots)$ from Λ_A put $\lambda x = (\lambda_\infty x_\infty, \lambda_2 x_2, \dots, \lambda_p x_p, \dots)$. It is obvious that the transformations $x \to xg$ and $x \to \lambda x$ commute with one another.

Consider the set $L_2(X)$ of all functions $f(x)$ on X with summable-

by a square. In this space there acts a unitary representation of the group G_A by the formula $T_0(g) f(x) = f(xg)$. Let us find the decomposition of this representation into irreducible representations.

Consider all possible characters $\pi(\lambda)$ on the group Λ_A such that $\pi(\lambda) = 1$ when $\lambda \in \Lambda$. To each such character we associate an irreducible representation $T_\pi(g)$ of the group G_A. This representation is constructed in the space of functions $f_\pi(x)$ on X satisfying the condition

$$f_\pi(\lambda x) = \pi(\lambda) |\lambda|^{-1} f(x), \quad |\lambda| = |\lambda_\infty|_\infty |\lambda_2|_2 \cdots |\lambda_p|_p \cdots,$$

and such that

$$\|f_\pi\|^2 = \int |f_\pi(x)|^2\, d\omega < +\infty,$$

where the integration is taken over the "circumference" Ω:

$$|x_\infty| = 1, \quad |x_2| = 1, \dots, \quad |x_p| = 1, \dots;$$

$d\omega$ is a measure on Ω defined in the natural way. It can be shown that two such representations $T_{\pi_1}(g)$ and $T_{\pi_2}(g)$ are equivalent if and only if either $\pi_1 = \pi_2$, or $\pi_1 = \bar{\pi}_2^{-1}$.

Theorem 2. *The representation in the space $L_2(X)$ decomposes into a direct sum of irreducible representations $T_\pi(g)$. This decomposition is given by the following formulas:*

$$f(x) = \int f_\pi(x)\, d\pi, \qquad \|f\|^2 = \int \|f_\pi\|^2\, d\pi,$$

where

$$f_\pi(x) = \int \pi^{-1}(\lambda)|\lambda|\, f(\lambda x)\, d\lambda.$$

Here $d\lambda$ is the invariant measure on the group Λ_A/Λ; $d\pi$ is the invariant measure on the character group of Λ_A/Λ.

5. We shall call **horospheres** in the space $Y = G_A/\Gamma$ the sets of the form yg_1zg_2, where $y \in Y$; g_1, g_2 are fixed elements of G_A, and z runs through the subgroup Z_A. It is easy to show that *every compact horosphere in Y has the form y_0zg, where y_0 is a fixed point corresponding to the identity adjacency class.* Hence it follows that the set of compact horospheres in Y is homogeneous and isomorphic to the space $X = G_A/Z_{AD}$.

Introduce a mapping $f(y) \xrightarrow{u} \varphi(x)$ from the space $L_2(Y)$ into the space of functions on X, defining it by the following formula

$$\varphi(x) = \int_{Z_A/Z} f(y_0zg)\, dz, \tag{1}$$

where g is a representative of the adjacency class corresponding to the point x.

Let $\mathcal{L}^{(0)}(Y)$ be the kernel of the mapping (1). Obviously, $\mathcal{L}^{(0)}(Y)$ is a closed invariant subspace of $L_2(Y)$. Denote by $\mathcal{L}^{(1)}(Y)$ the orthogonal complement to the space $\mathcal{L}^{(0)}(Y) + \mathcal{L}_c$, where $\mathcal{L}_c$ consists of the functions constant on Y.

Theorem 3. *The space $\mathcal{L}^{(0)}(Y)$ decomposes into a sum of a countable number of irreducible representations of the group G_A. The space $\mathcal{L}^{(1)}(Y)$ decomposes into a continuous sum of irreducible representations entering into $L_2(X)$, and contains each of them once.*

Proof. The first assertion is proved by the methods developed in (4). We outline the proof of the second assertion. Denote by $\mathcal{L}'(X)$ the totality of functions on

X arising from the mapping (1). Let $\varphi(x)$ be an arbitrary finite function on X satisfying the following condition: there exists a $p = p_0$ such that $\varphi(xg) = \varphi(x)$ for all adeles g of the form

$$g = (1, \dots, 1, u_{p_0}, \dots, u_q, \dots), \qquad u_q \in U_q.$$

Consider the mapping $\varphi(x) \xrightarrow{v} f(y)$ of the function $\varphi(x)$ into the space of functions on Y:

$$f(y) = \sum_{\gamma \in G/ZD} \varphi^*(\gamma g). \tag{2}$$

---

* It is also useful to note that the irreducible representations entering into the decomposition of $\mathcal{L}^1(Y)$ split into a countable number of series, the representations of each series being determined by one continuous parameter. Meanwhile, arbitrary irreducible representations of the adele group depend on a countable number of continuous parameters.

Here $\varphi^*(g)$ is the natural extension of the function $\varphi(x)$ to the whole group G_A, and the summation is over the set of right cosets G/ZD. It can be shown that the composition $M = uv$ of the mappings v and u takes finite functions $\varphi(x)$ to functions in $\mathcal{L}'(X)$ and is given by the following formula:

$$M\varphi(x) = \varphi(x) + \int_{Z_A} \varphi^*(szg)\, dz,$$

where

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The operator M commutes with the representation, and therefore maps each of the pair of equivalent irreducible representations T_π and $T_{\pi^{-1}}$ into their direct sum. Thus it is given on the sum of these representations by some matrix of the second order. We shall show that this matrix is degenerate, i.e. the images of the representations T_π and $T_{\pi^{-1}}$ coincide; from this the second assertion of the theorem follows easily. To this end we choose, in a canonical way, in the spaces of the representations T_π and $T_{\pi^{-1}}$, the vectors φ_π and $\varphi_{\pi^{-1}}$. Namely, let $\pi = (\pi_\infty, \pi_2, \dots, \pi_p, \dots)$, where $\pi_p(t)$ is a multiplicative character in the field of p-adic numbers Q_p; extend $\pi_p(t)$ to a character in the quadratic extension $Q_p(\sqrt{\varepsilon_p})$ of the field Q_p, where $|\varepsilon_p|_p = 1$. Let $x = (x_\infty, x_2, \dots, x_p, \dots)$, $x_p = (x_p^1, x_p^2)$, be a point of the space X. Then we put

$$\varphi_\pi(x) = \prod_p \pi_p(x_p^1 + \sqrt{\varepsilon_p}\, x_p^2)\, |x_p^1 + \sqrt{\varepsilon_p}\, x_p^2|_p^{-1},$$

$$\varphi_{\pi^{-1}}(x) = \prod_p \pi_p(x_p^1 - \sqrt{\varepsilon_p}\, x_p^2)\, |x_p^1 - \sqrt{\varepsilon_p}\, x_p^2|_p^{-1}.$$

It can be shown that $M\varphi_\pi = \varphi_\pi + \alpha(\pi)\varphi_{\pi^{-1}}$, $M\varphi_{\pi^{-1}} = \beta(\pi)\varphi_\pi + \varphi_{\pi^{-1}}$. Here the multipliers $\alpha(\pi)$ and $\beta(\pi)$ are expressed in terms of the B-function and the Dirichlet L-function $L(s,\chi)$ by the following formulas:

$$\alpha(\pi) = B(-s/2,\, 1/2)\, L(-s,\chi)\, L^{-1}(1-s,\chi)\, k^{-1/2}\sigma,$$

$$\beta(\pi) = B(s/2,\, 1/2)\, L(s,\overline{\chi})\, L^{-1}(1+s,\overline{\chi})\, k^{-1/2}\sigma.$$

Here s is a complex number, χ is the multiplicative character on the set of natural numbers determined by the character π; k is the period of the character χ; $\sigma\overline{\sigma} = 1$. On the basis of the known functional equation for Dirichlet L-functions we obtain that $\alpha(\pi)\beta(\pi) = 1$, and hence the matrix

$$\begin{pmatrix} 1 & \alpha(\pi) \\ \beta(\pi) & 1 \end{pmatrix}$$

is degenerate.

Received
8 I 1964

REFERENCES

1. A. Weil, *Adels and Algebraic Groups*, N. Y., 1961.

2. A. Borel, Some Finiteness Properties of the Adele Groups over Number Fields, Paris, 1963.

3. I. M. Gel' fand, DAN, 70, No. 1, 5 (1950).

4. I. M. Gel' fand, I. I. Pyatetskii-Shapiro, Tr. Moscow Math. Soc., 12, 389 (1963).

5. N. G. Chudakov, *Introduction to the Theory of Dirichlet L-functions*, 1947.

Note: Figure translations are in progress. See original paper for figures.