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# MATHEMATICS

V. V. PETROV

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**Abstract**

**Full Text**

MATHEMATICS

V. V. PETROV

## ON THE PROBABILITIES OF LARGE DEVIATIONS OF SUMS OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES

*(Presented by Academician V. I. Smirnov on 30 IX 1963)*

0. Let  $X_1, X_2, \dots$  be a sequence of independent random variables having the same nondegenerate distribution. The present paper contains limit theorems for the probabilities of large deviations of the sum  $X_1 + X_2 + \dots + X_n$  in the cases of a nonlattice and a lattice distribution of the variable  $X_1$ . Each of these theorems includes the condition that the moment-generating function  $Ee^{hX_1}$  be finite in some nondegenerate interval one of whose endpoints is the point  $h = 0$ . The existence for the random variable  $X_1$  of a finite variance or a finite mathematical expectation is nowhere assumed, with the exception of Theorem 5.
1. We first formulate some definitions and auxiliary results. Denote by  $V(x)$  the distribution function (d.f.) of the random variable  $X_1$ , so that  $V(x) = P(X_1 < x)$ . A point  $x$  is called a **point of increase** of the d.f.  $V(x)$  if  $V(x + \varepsilon) - V(x - \varepsilon) > 0$  for any  $\varepsilon > 0$ . The set of points of increase of the d.f.  $V(x)$  will be denoted by  $\mathfrak{A}$ . Put

$$A = \sup \mathfrak{A}. \tag{1}$$

By  $\mathfrak{B}^+$  denote the set of all those nonnegative values  $h$  for which

$$\int_0^{+\infty} e^{hx} dV(x) < \infty. \tag{2}$$

This set contains at least one point,  $h = 0$ . Put

$$B = \sup \mathfrak{B}^+. \tag{3}$$

It is clear that if  $A < +\infty$ , then  $B = +\infty$ . In the case where  $A = +\infty$ , either  $B = +\infty$  or  $B < +\infty$  may occur.

**Lemma.** Let  $B > 0$ . Put, for  $0 < h < B$ ,

$$R(h) = \int_{-\infty}^{\infty} e^{hx} dV(x); \quad (4)$$

$$m(h) = \frac{1}{R(h)} \int_{-\infty}^{\infty} x e^{hx} dV(x); \quad (5)$$

$$\sigma^2(h) = \frac{dm(h)}{dh}. \quad (6)$$

The following assertions are true:

I.  $0 < \sigma^2(h) < +\infty$  for  $0 < h < B$ .

II. The function  $m(h)$  is strictly increasing and continuous in the interval  $0 < h < B$ .

III.  $-\infty \leq EX_1 < +\infty$ ;  $\lim_{h \downarrow 0} m(h) = EX_1$ .

IV. There exists the limit

$$\lim_{h \uparrow B} m(h) = A_0. \quad (7)$$

Here  $A_0 = A$ , if  $A < +\infty$  or if  $A = +\infty$  and at the same time  $B = +\infty$ . If, however,  $A = +\infty$ , but  $B < +\infty$ , then the case  $A_0 < +\infty$ , i.e.  $A_0 \neq A$ , is possible.

V. Whatever the number  $y$  from the interval  $EX_1 < y < A_0$ , the equation  $m(h) = y$  has a unique real root  $h^*$ . Moreover,  $0 < h^* < B$ .

The assertions of the lemma can be extracted from the works of H. Cramér <sup>(1)</sup>, H. Chernoff <sup>(2)</sup>, and H. Daniels <sup>(3)</sup>.

**2. Theorem 1.** Let  $X_1, X_2, \dots$  be a sequence of independent random variables having the same nonlattice distribution, such that  $B > 0$ . Suppose, further,

$$EX_1 > -\infty, \quad A_0 < +\infty, \quad (8)$$

where  $A_0$  is defined by equality (7). Then

$$\mathbf{P}(X_1 + \dots + X_n \geq nx) = \frac{\exp\{n \ln R(h) - nhx\}}{h\sigma(h)\sqrt{2\pi n}} (1 + o(1)) \quad (9)$$

as  $n \rightarrow \infty$ , uniformly with respect to  $x$  in the region

$$EX_1 + \varepsilon \leq x \leq A_0 - \varepsilon, \quad (10)$$

where  $\varepsilon$  is an arbitrarily small positive constant. Here  $h$  is the unique real root of the equation  $m(h) = x$ .

**Remark.** If, instead of condition (8), one of the conditions

$$EX_1 = -\infty, \quad A_0 < +\infty, \quad (11)$$

$$EX_1 = -\infty, \quad A_0 = +\infty, \quad (12)$$

$$EX_1 > -\infty, \quad A_0 = +\infty, \quad (13)$$

is satisfied, then Theorem 1 remains valid with region (10) replaced respectively by the regions

$$-C_1 \leq x \leq A_0 - \varepsilon, \quad (14)$$

$$-C_1 \leq x \leq C_2, \quad (15)$$

$$EX_1 + \varepsilon \leq x \leq C_2, \quad (16)$$

where  $C_1$  and  $C_2$  are arbitrary large positive constants.

A relation of type (9) was first obtained by Cramér<sup>(1)</sup> for a narrower class of distributions. In<sup>(1)</sup>, instead of the condition  $B > 0$ , which imposes a restriction on the behavior of the distribution function  $V(x)$  of the random variable  $X_1$  only on the positive half-axis  $0 < x < +\infty$ , the condition was assumed to hold:  $Ee^{hX_1} < \infty$  for  $-a < h < a$  and for some  $a > 0$ .

The following theorem follows from Theorem 1.

**Theorem 2.** Let the conditions of Theorem 1 be satisfied, and let  $c$  be an arbitrary constant from the interval  $EX_1 < c < A_0$  and  $\delta(n)$  an arbitrary function satisfying the condition  $\lim_{n \rightarrow \infty} \delta(n) = 0$ . Then

$$\begin{aligned} \mathbf{P}(X_1 + \dots + X_n \geq n(c + \alpha_n)) &= (h\sigma(h)\sqrt{2\pi n})^{-1} \times \\ &\times \exp \left\{ n \left[ \ln R(h) - h(c + \alpha_n) - \frac{\alpha_n^2}{2\sigma^2(h)} (1 + O(|\alpha_n|)) \right] \right\} (1 + o(1)) \end{aligned} \quad (17)$$

as  $n \rightarrow \infty$ , uniformly with respect to  $c$  and  $\alpha_n$  in the regions

$$EX_1 + \varepsilon \leq c \leq A_0 - \varepsilon, \quad (18)$$

$$|\alpha_n| \leq \delta(n), \quad (19)$$

where  $\varepsilon$  is an arbitrarily small positive constant.\* Here  $h$  is the unique real root of the equation  $m(h) = c$ .

\* If, instead of (8), one of the conditions (11)–(13) is satisfied, then (18) should be replaced by the corresponding region from (14)–(16).

This theorem is a generalization of a theorem of R. L. Dobrushin (1974), who considered the case in which the random variable  $X_1$  has a finite number of possible values.

In turn, from Theorem 2 we obtain the following corollary:

**Theorem 3\*.** *Let  $X_1, X_2, \dots$  be a sequence of independent random variables having the same nonlattice distribution such that  $B > 0$ . Let  $c$  be any constant satisfying the condition  $EX_1 < c < A_0$ . Then*

$$\mathbf{P}(X_1 + \dots + X_n \geq nc) = \frac{\exp\{n \ln R(h) - nhc\}}{h\sigma(h)\sqrt{2\pi n}}(1 + o(1))$$

as  $n \rightarrow \infty$ . Here  $h$  is the unique real root of the equation  $m(h) = c$ .

**Theorem 4.** *If the conditions of Theorem 1 are fulfilled and, in addition, the characteristic function  $V(x)$  of the random variable  $X_1$  has a nonzero absolutely continuous component, then the assertion of Theorem 1 remains valid with the remainder term  $o(1)$  in (9) replaced by  $O(\frac{1}{n})$ .*

The introduction of the additional condition indicated in Theorem 4 makes it possible to replace  $o(1)$  by  $O(\frac{1}{n})$  in Theorem 3 as well. A further improvement of the order of the remainder term  $O(\frac{1}{n})$  is impossible without introducing new restrictions.

**Theorem 5.** *If the conditions of Theorem 1 are fulfilled and if, in addition,  $E(X_1)^3 < \infty$ , then relation (9) holds as  $n \rightarrow \infty$  and*

$$EX_1 + \frac{\rho(n)}{\sqrt{n}} \leq x \leq A_0 - \varepsilon,$$

whatever the constant  $\varepsilon > 0$  and the function  $\rho(n)$  satisfying the condition

$$\lim_{n \rightarrow \infty} \rho(n) = +\infty.$$

Here  $h$  is the unique root of the equation  $m(h) = x$ .

3. Let us now consider the case where the random variable  $X_1$  has a lattice distribution.

**Theorem 6.** *Let  $X_1$  take, with positive probabilities, only values of the form  $a + NH$  ( $N = 0, \pm 1, \pm 2, \dots$ ), where  $a$  is some fixed real number and  $H$  is the maximal span of the distribution. Suppose, further, that conditions (8) and  $B > 0$  are fulfilled. If  $nx$  takes only values of the form  $na + NH$  ( $N$  an integer), then the following assertions are valid:*

$$\text{I. } \mathbf{P}(X_1 + \dots + X_n = nx) = \frac{\exp\{n \ln R(h) - nhx\}}{\sigma(h) \frac{1}{H} \sqrt{2\pi n}} \left(1 + O\left(\frac{1}{n}\right)\right)$$

as  $n \rightarrow \infty$ , uniformly with respect to  $x$  in the domain (10).

$$\text{II. } \mathbf{P}(X_1 + \dots + X_n \geq nx) = \frac{H \exp\{n \ln R(h) - nhx\}}{\sigma(h) \sqrt{2\pi n} (1 - e^{-Hh})} \left(1 + O\left(\frac{1}{n}\right)\right)$$

as  $n \rightarrow \infty$ , uniformly with respect to  $x$  in the domain (10).

Here  $\varepsilon$  is an arbitrarily small positive constant, and  $h$  is the unique real root of the equation  $m(h) = x$ .

For Theorem 6 the remark made in connection with Theorem 1 remains in force. From Theorem 6 there follow corollaries analogous to Theorems 2 and 3.

Under stronger assumptions, analogous results were obtained by D. Blackwell and J. Hodges (<sup>6</sup>) and by R. L. Dobrushin (<sup>4</sup>). Other local limit theorems for large deviations are contained in the works of H. Daniels (<sup>3</sup>) and V. M. Zolotarev (<sup>7</sup>).

\* As became known to the author of the present paper only after obtaining all the results contained in it, Theorem 3 had been obtained (in a different formulation) by Bahadur and Ranga Rao (<sup>5</sup>).

4. The proofs of the theorems formulated are carried out according to the same plan as the proofs of Cramér's theorems<sup>1</sup>. In doing so, essential use is made of the lemma and of known refinements of limit theorems of the classical type.

In conclusion, let us note that, in the same way, one can formulate results concerning the asymptotic behavior of the probability  $P(X_1 + \dots + X_n < nx)$ . In this case, instead of the condition  $B > 0$ , one should impose the condition  $b < 0$ , where  $b = \inf \mathfrak{B}^-$ , and  $\mathfrak{B}^-$  is the set of all nonpositive values  $h$  for which

$$\int_{-\infty}^0 e^{hx} dV(x) < \infty.$$

<sup>1</sup>H. Cramér, *Actual. sci. et ind.*, No. 736, Paris, 1938.

If  $b < 0$ , then  $-\infty < EX_1 \leq +\infty$ , and for  $b < h < 0$  one may consider the functions  $R(h)$ ,  $m(h)$ , and  $\sigma^2(h)$ , defined by the equalities (4)–(6). The admissible values of  $x$  in the analogue of Theorem 1 will then be the values in the range  $a_0 + \varepsilon \leq x \leq EX_1 - \varepsilon$ , where  $a = \lim_{h \downarrow b} m(h)$ .

Leningrad State University  
named after A. A. Zhdanov

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## REFERENCES

*Note: Figure translations are in progress. See original paper for figures.*

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